An uniform ergodic theorem for linearly repetitive sets

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Fibonacci sequence

Let $\mathbf{x}_{Fib} = (x_n)_{n\in\mathbb{Z}}$ be the Fibonacci sequence.

substitution: $a \mapsto ab$, $b \mapsto a$

$$b.a$$

$$a.ab$$

$$ab.aba$$

$$\vdots$$

$$\dots abaab.abaababa \dots$$

$X = \overline{\{\sigma^n(\mathbf{x}_{Fib})\}}$ is uniquely ergodic.

Let $w = x_{[n,n+k]}$, $n \in \mathbb{Z}$ and $k > 0$.

Then,

$$\frac{1}{2N}\sharp\{\ell \in [-N, N] \mid x_{[\ell,\ell+k]} = w\}$$

converges to *frequency* of w, but how fast? Last equation is just the Birkhoff Ergodic theorem for σ and $\chi_{[.w]}$.

General Cohomological Considerations

Let (X, σ) be a uniquely ergodic shift (like before), and f be continuous function. Then

$$\frac{1}{N}\sum_{i=0}^{N-1} f(\sigma^i(x)) \overset{N\to+\infty}{\to} \int f \,\mathrm{d}\mu$$

uniformly in $x \in X$, i.e.

$$\mathrm{dev}_f(x, N) := \sum_{i=0}^{N-1} f(\sigma^i(x)) - N\int f \,\mathrm{d}\mu$$

is sublinear.

Theorem (Halasz,Kachurovskii)

The best rate of convergence is $\mathrm{dev}_f(x, N)$ *uniformly bounded and this happens when* f *is* $\mathrm{dev}_f(x, N)$ *is uniformly bounded if and only if* dev *is a continuous coboundary,i.e., there exists* ϕ *such that*

$$f(x) - \int f \,\mathrm{d}\mu = \phi(\sigma(x)) - \phi(x)$$

Cohomological Considerations

If $f = \chi_C$, then

Corollary (Halasz)

If $\frac{1}{2N}\sharp\{\ell \in [-N, N] \mid x_{[\ell,\ell+k]} = w\} = O(1/N)$*, then* $\mu([w])$ *is an eigenvalue of* (X, σ)*.*

$$a \mapsto \psi(a) = aba,$$
$$b \mapsto \psi(b) = ab.$$

$$M = \begin{array}{c|cc} & a & b \\ \psi(a) & 2 & 1 \\ \psi(b) & 1 & 1 \end{array}$$

$freq(a)$:

$$\begin{array}{rc} 1/2 & (b.a) \\ 3/5 & (\psi(b.a)) \\ 8/13 & (\psi^2(b.a)) \\ 21/34 & (\psi^3(b.a)) \\ \vdots & \end{array}$$

$$F_{2n+2}/F_{2n+3}$$
$$\downarrow$$
$$1/\varphi, \quad \varphi = \tfrac{\sqrt{5}+1}{2}$$

$$b.a$$
$$ab.aba$$
$$abaab.abaababa$$
$$abaababaabaab.abaababaabaababaababa$$

$$F_n = \frac{1}{\sqrt{5}}(\varphi^n - (-1/\varphi)^n)$$

Then

$$F_{2n+2} - F_{2n+3}\frac{1}{\varphi} = -\frac{1}{\sqrt{5}}(1+\varphi^2)\varphi^{-2n-4}$$

Convergence is exponential on n!
But the size of the window grows exponentially!

$|a|b|a|a|b|a|b|a|a|b|a|a|b|.a|b|a|a|b|a|b|a|a|b|a|a|b|a|b|a|a|b|a|b|a|$

$|a\ b\ a|a\ b|a\ b\ a|a\ b\ a|a\ b|.a\ b\ a|a\ b|a\ b\ a|a\ b\ a|a\ b|a\ b\ a|a\ b|a\ b\ a|$

$|a\ b\ a\ a\ b\ a\ b\ a|a\ b\ a\ a\ b|.a\ b\ a\ a\ b\ a\ b\ a|a\ b\ a\ a\ b|a\ b\ a\ a\ b\ a\ b\ a|$

$|a\ b\ a\ a\ b\ a\ b\ a\ a\ b\ a\ a\ b|.a\ b\ a\ a\ b\ a\ b\ a\ a\ b\ a\ a\ b\ a\ b\ a\ a\ b\ a\ b\ a$

How to use the previous convergence and decomposition to estimate the deviation?

Example: $N = 12$, need to count how many a's in

$$baababaabaab.abaababaabaa$$

Let $n = 2 \sim \lfloor \log N \rfloor$. To the right we have

$$.\overbrace{\underbrace{a\ b\ a\ a\ b\ a\ b\ a}_{\psi^2(a)}\ \underbrace{a\ b\ a}_{\psi(a)}\ a|b\ a\ b\ a\ a\ b\ a\ b\ a}^{\psi^3(a)}$$

Thus,

$$n_a(\mathbf{x}_{[0,12]}) = n_a(\psi^2(a)) + \qquad n_a(\psi(a)) + 1$$

How to use the previous convergence and decomposition to estimate the deviation?
Example: $N = 17$, need to count how many a's in

$$baababaabaab.abaababaabaababaa$$

Let $n = 2 \sim \lfloor \log N \rfloor$. To the right we have

$$.\overbrace{\underbrace{a\ b\ a\ a\ b\ a\ b\ a}_{\psi^2(a)}\ \underbrace{a\ b\ a\ a\ b}_{\psi^2(b)}\underbrace{a\ b\ a}_{\psi(a)}\ a}^{\psi^3(a)}|b\ a\ b\ a$$

Thus,

$$n_a(\mathbf{x}_{[0,17]}) = n_a(\psi^2(a)) + n_a(\psi^2(b)) + n_a(\psi(a)) + 1$$

More generally,

$$n_a(\mathbf{x}_{[0,N]}) - \frac{N}{\varphi} = \sum_{k=0}^{\lfloor \log N \rfloor} \ell_{a,k}\Big(n_a(\psi^k(a)) - \frac{|\psi^k(a)|}{\varphi}\Big) + \sum_{k=0}^{\lfloor \log N \rfloor} \ell_{b,k}\Big(n_a(\psi^k(b)) - \frac{|\psi^k(b)|}{\varphi}\Big) \quad (0.1)$$

$$\ell_{a,k} < C, n_a(\psi^k(a)) = F_{2k-1}, \quad |\psi^k(a)| = F_{2k}$$

and similarly for b. Hence,

$$\|n_a(\mathbf{x}_{[0,N]}) - \frac{N}{\varphi}\| \leq 2C\frac{1+\varphi^2}{\sqrt{5}\varphi^3} \sum_{k=0}^{\lfloor \log N \rfloor} \varphi^{-2n} \leq C'$$

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All of these formulas work since $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is a Pisot Matrix!

Of course, we get again that $1/\varphi$ is an eigenvalue of (X_{fib}, σ).

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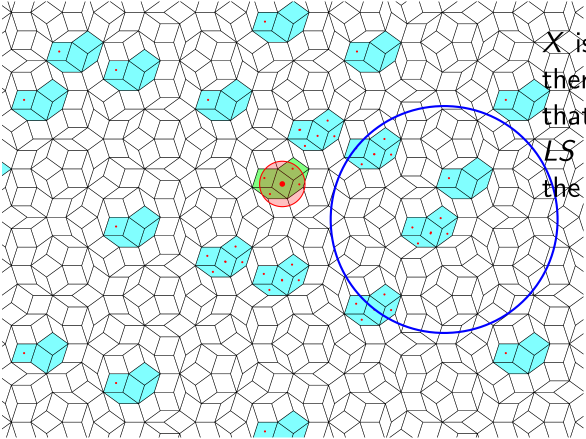
Theorem

If (X, σ) is associated to a Pisot substitution, then there is fast convergence (for all words)!

Linearly repetitivity

For X a tiling or a Delone set

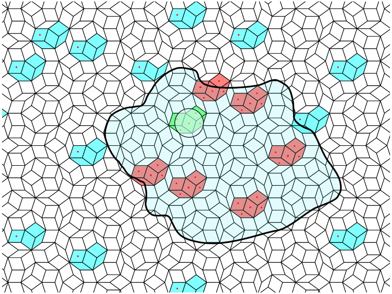


X is linearly repetitive if there exists $L > 1$ such that every ball of radius LS contains a copy of all the S-patches in X.

Uniform patch frequencies

Given a bounded measurable set $D \subset \mathbb{R}^d$ and $\mathbf{p} = X \cap B_r(x_0)$, define

$$n_{\mathbf{p}}(D) = \text{card}\{y \in Y \cap D \mid Y \cap \overline{B}_r(y) \text{ is a copy of } \mathbf{p}\}.$$



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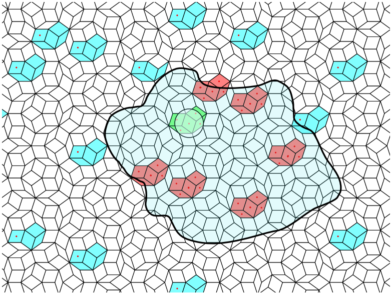


Definition

X has *uniform patch frequencies* if

$$\lim_{N \to +\infty} \frac{n_{\mathbf{p}}([-N, N]^d + x)}{\mathrm{vol}([-N, N]^d + x)} = \mathrm{freq}(\mathbf{p})$$

exists uniformly on $x \in \mathbb{R}^d$.

Lagarias and Pleasants's Theorem

The deviation of frequence for a patch $\mathbf{p}$ in a measurable set E is defined:

$$\mathrm{dev}_p(E) = |n_{\mathbf{p}}(E, X) - \mathrm{freq}(\mathbf{p})\,\mathrm{vol}(E)|. \tag{0.2}$$

Theorem (Lagarias and Pleasants)

Let X be a linearly repetitive tiling. Let E_N be either the sequence of balls $B_N(0)$ or the sequence of boxes $[-N, N]^d$. There exists $\delta > 0$ such that for every patch $\mathbf{p}$ of X we have

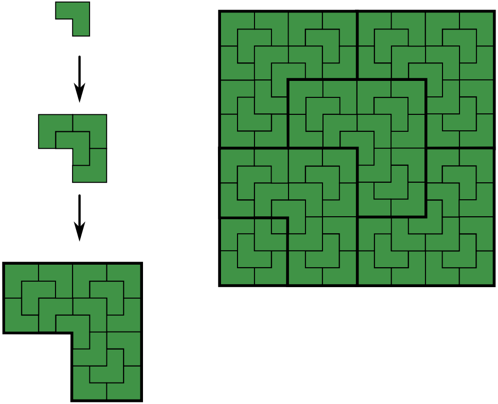
$$\mathrm{dev}_p(E_N) = O(N^{d-\delta}), \tag{0.3}$$

Remark

In the original proof, δ depends on E_N.

Theorem

Linearly repetitive tilings admit a "properly nested" sequence of tilings (conjugate to the original one).



Remark

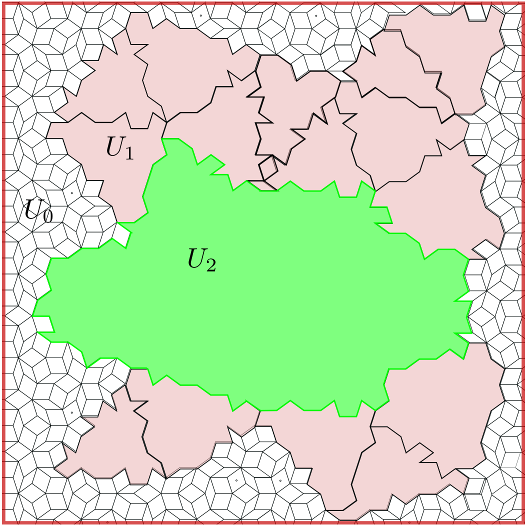
This sequence is not stationary, unless the tiling is self-similar, but in this case, the sequence is obvios and DO not follow from our construction.

Lemma

This properly nested sequence of tilings induces a Non-stationary Markov Chain such that counting the occurrences of a tile of level m in a tile of level n converges exponentially fast to the frequency of the tile.

Proposition

Each region D can be “hierarchically decomposed”



Theorem (Coronel, A.-P.)

Let X be linearly repetitive. There is $\delta > 0$ (which depends only on X) such that

$$\mathrm{dev}_{\mathbf{p}}(E_N)/\operatorname{vol}(E_N) = O(N^{-\delta})$$

Remark

E_N may be $[-N, N]^d$, $B_N(0)$ or $E_N = N \cdot E$, where E is a convex set, more general?

Theorem

If X is self-similar (Primitive Matrix) with Perron Eigenvalue λ and second eigenvalue τ. Let $\rho = \tau\lambda^{1/d-1}$. Then:

- *if $\rho < 1$ then convergence is fast!*
- *if $\rho = 1$ then $\mathrm{dev}_p(E_N)/vol E_N = O(\log^m N)$.*
- *if $\rho > 1$ then $\mathrm{dev}_p(E_N)/vol E_N = O(N^{-\delta})$.*