Tower systems for Linearly Repetitive Delone sets

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Workshop for Aperiodic order and dynamics
Bielefeld, March 2011

Motivation

Let X be an aperiodic (general) repetitive Delone set (or tiling) and let (Ω, T) be its associated tiling dynamical system: We want to understand the dynamical properties of the translation action T:

- Eigenvalues?
- Ergodic/Mixing properties?

The idea of this talk is to introduce a technique, that allows to understand these kind of problems, and hopefully others.

Minimal Cantor systems

- A minimal Cantor system (X, T) is a pair s.t.:
  - X is a Cantor set.
  - $T : X \to X$ is a minimal homeomorphism, i.e., every orbit is dense.
  - Fix μ an T-invariant probability measure.
- λ in $\mathcal{S}^1$ is an eigenvalue of (X, T) if there exists $f \in L^2(X, \mathcal{S}^1)$ such that

$$f(Tx) = \lambda f(x)$$

  for all $x \in X$.
- λ is a continuous eigenvalue if f can be chosen continuous.

Question

How about studying the eigenvalues of (X, T)?

Kakutani-Rokhlin (KR) partitions

- A Rokhlin tower is a pairwise disjoint family $\mathcal{T}$ of measurable sets of the form
$$\mathcal{T} = \{T^j C\}_{j=0}^{h-1},$$
- Define *floor*, *height*, *stages* and give basic example.
- A Kakutani-Rokhlin partition $\mathcal{P}$ of X is a partition by Rohlin towers, i.e.,
$$\mathcal{P} = \{T^j C_i \mid i \in \{1, \ldots, n\}, j \in \{0, \ldots, h_i\}\},$$
- Define *base*.

Constructing KR partitions

Lemma (Host-Putnam-Skau?)

Let (X, T) be a minimal Cantor set. If C is any clopen-open subset of X, then there exists a KR partition $\mathcal{P}$ with base C.

Proof.

1. $R : C \to \mathbb{N}$ defined by
$$R(x) = \inf\{k > 0 \mid T^k(x) \in C\}$$
is continuous.
2. Hence, $R(C) = \{h_1, \ldots, h_k\}$ for some $k \in \mathbb{N}$.
3. Thus, setting $C_i := R^{-1}(h_i)$ we get a KR partition
$$\{T^j C_i : i \in \{1, \ldots, k\}, j \in \{0, \ldots, h_i - 1\}\}$$

□

Tower systems for minimal Cantor systems

A Kakutani-Rokhlin tower system for (X, T) is a sequence $(\mathcal{P}_n)_{n\in\mathbb{N}}$ of Kakutani-Rokhlin partitions such that:

- $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for all $n \in \mathbb{N}$.
- The base of $\mathcal{P}_{n+1}$ is included in the base of $\mathcal{P}_n$.
- Other technical conditions ...

Theorem (Host-Putnam-Skau?)

Every minimal Cantor system (X, T) admits a Kakutani-Rokhlin tower system.

$$\mathcal{P}_n = \{T^j B_i(n) \mid i \in \{1, \ldots, k(n)\}, j \in \{0, \ldots, h_i(n)\}\},$$

Proof.

Apply Lemma to a decreasing sequence $(C_n)_{n\in\mathbb{N}}$ of clopen subsets of X with $\mathrm{diam}(C_n) \to 0$ as $n \to \infty$. □

Characterization of Eigenvalues for minimal Cantor systems

Let (X, T) be a minimal Cantor system and $(\mathcal{P}_k)_k$ be a KR tower system. There exist "transition matrices" $M(n)$ such that

$$M_{l,k}(n) = \sharp\{0 \leq j < h_l(n) \mid T^j B_l(n) \subseteq B_k(n-1)\}.$$

Theorem (Cortez,Durand,Host, Maass, Bressaud)

Suppose that the matrices $M(n)$ are uniformly bounded (in size and norm). Let $\lambda = \exp(2\pi\alpha)$, where $\alpha \in \mathbb{R}$. Then:

- (X, T) *is uniquely ergodic.*
- (X, T, μ) *is not strongly mixing.*
- λ *is an eigenvalue if and only if* $\sum_{n\geq 0} \max_k |\lambda^{h_k(n)} - 1|^2 < +\infty$.
- λ *is a continuous eigenvalue if and only if* $\sum_{n\geq 0} \max_k |\lambda^{h_k(n)} - 1| < +\infty$.

Motivation II

Let X be an aperiodic (general) repetitive Delone set (or tiling) and let (Ω, T) be its associated tiling dynamical system: We want to understand the dynamical properties of the translation action T:

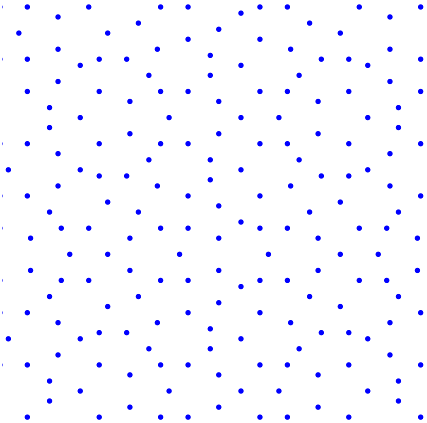
Question

Which of these results still hold for tilings and delone sets?

Question

What's the relation between minimal Cantor sets and Tilings and Delone sets.

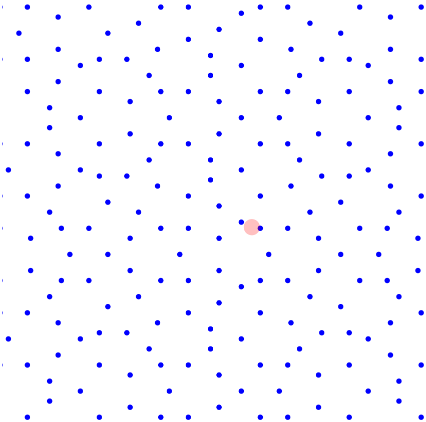
Delone sets



A subset X of the Euclidean space $\mathbb{R}^d$ is *Delone* if

- it is uniformly discrete,
- and relatively dense.

Delone sets



A subset X of the Euclidean space $\mathbb{R}^d$ is *Delone* if

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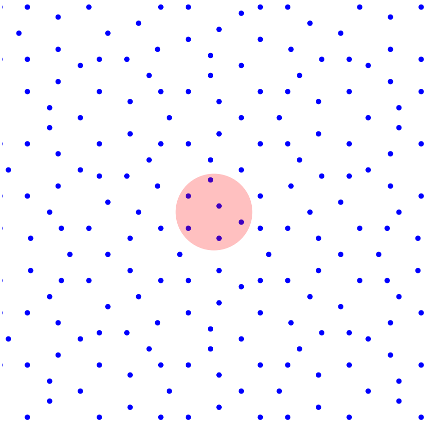
Delone sets



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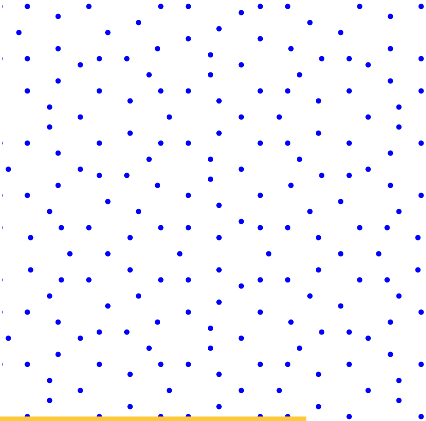
There exists $R > 0$ s.t. every ball of radius R contains *at least* one point of X.

Repetitive Delone sets

Let X be a Delone set. Given $S > 0$ and $x \in X$, the *S-pattern* around x is defined as

$$X \wedge B(x, S) := (X \cap B(x, S), B(x, S)).$$



- We say that X is repetitive if for every $S > 0$ there exists $M > 0$ such that each ball of radius M contains a *translated copy* of every S-patch of X.
- The repetitivity function $M_X(S)$ is the smallest such M.
- X is linearly repetitive if there exists $L > 1$ such that $M_X(S) \leq LS$.

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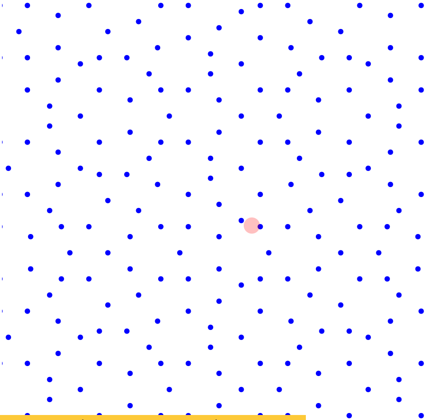

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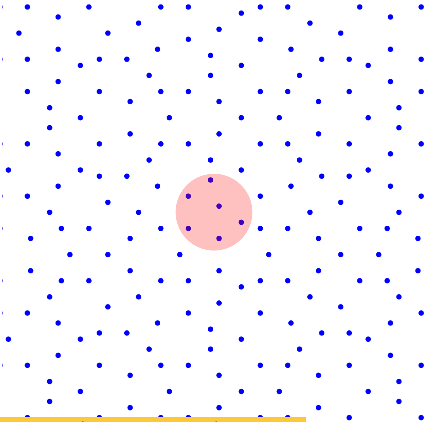

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Delone dynamical systems

Let X be a repetitive Delone set.

- Given $t \in \mathbb{R}^d$ define

$$T_t X := X - t = \{x - t : x \in X\}.$$

- X is **aperiodic** if $T_t X \neq X$ for all $t \in \mathbb{R}^d$.
- The T-orbit of X is

$$X - \mathbb{R}^d = \{X - t : t \in \mathbb{R}^d\}.$$

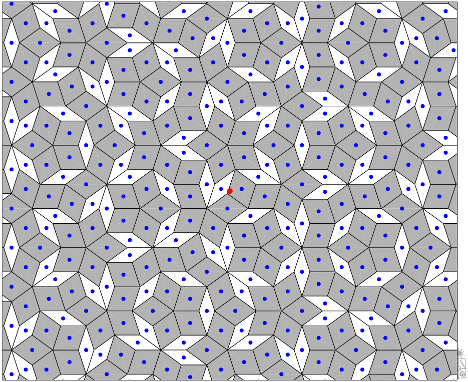
- $\mathrm{dist}(X - t, X - s) < \epsilon$ if

$$X \cap B_R(t) \equiv X \cap B_R(s + x),$$

where $\|x\| < \epsilon$ and $R > 1/\epsilon$.

Definition

The *hull* Ω is the completion of $X - \mathbb{R}^d$.



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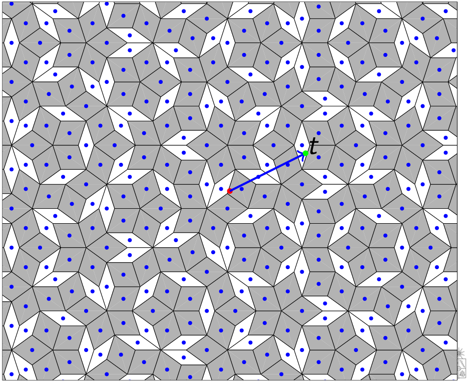

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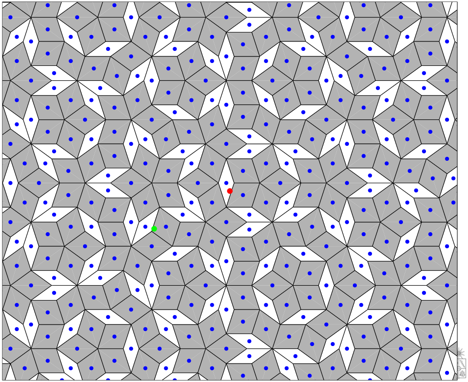

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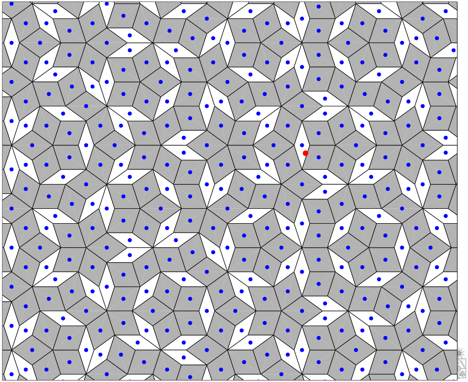

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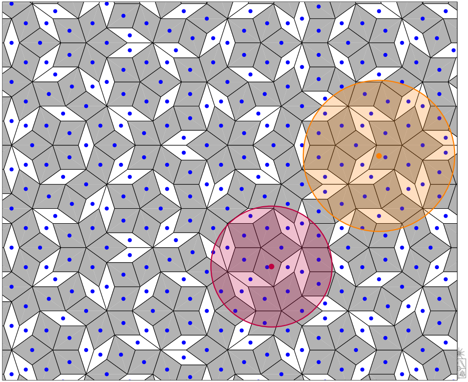

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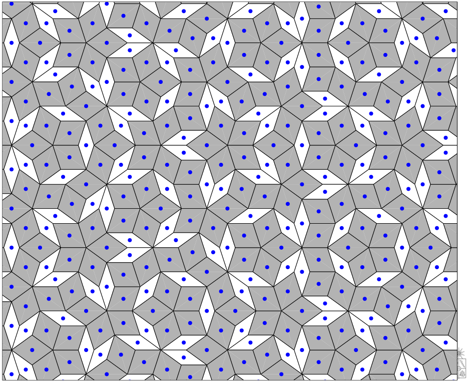
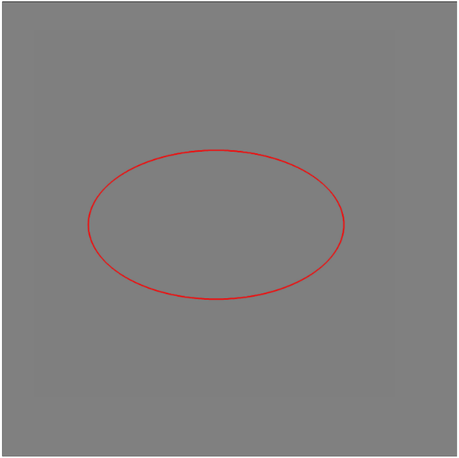


Toy examples of Delone dynamical systems

If $X = \mathbb{Z}$, then:

- $X - k = X$ for all $k \in \mathbb{Z}$.
- $\Omega = \mathbb{R}/\mathbb{Z}$.



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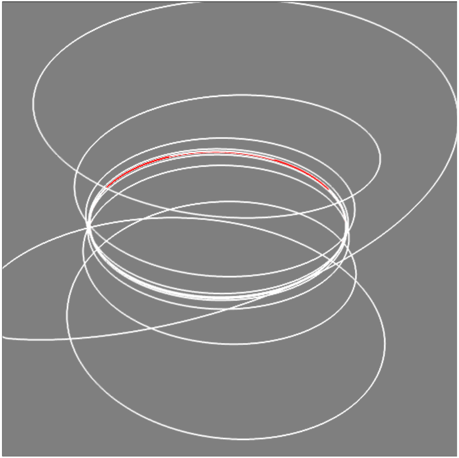
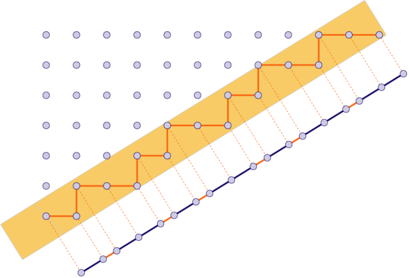
- $\Omega = \mathbb{R}/\mathbb{Z}$.

If $X = \mathbb{Z} \setminus \{0\}$, then:

- $\mathbb{Z}$ belongs to Ω.
- Ω has two path components.
  - $X - \mathbb{R}$ (in white).
  - $\mathbb{Z} - \mathbb{R}$ (in red).



Fibonacci: Model example in $d = 1$



If 0 is a vertex of a tiling Y in Ω, then Y can be coded.

Example

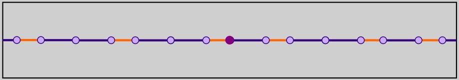


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Figure: A picture of the Hull

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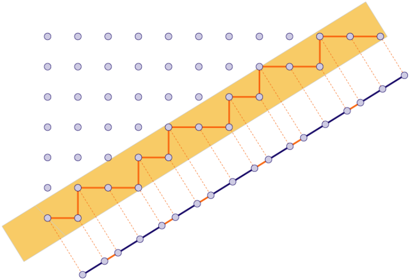

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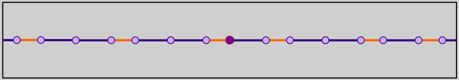

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Figure: A picture of the Hull

Dynamical systems over the Hull

Let X be an aperiodic repetitive Delone set. There is a natural dynamical system over the hull Ω:

- The translation action $T : \Omega \times \mathbb{R} \to \Omega$ defined by

$$T_t(Y) = Y - t,$$

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- The translation action $T : \Omega \times \mathbb{R} \to \Omega$ defined by

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- this action is continuous,
- moreover, (Ω, T) is minimal (since X is repetitive),
- and it gives much information about the structure of X.

The Canonical transversal

Let X be an aperiodic repetitive Delone set and (Ω, T) be its dynamical system: The canonical transversal is defined by

$$\Omega_0 = \{Y \in \Omega \mid 0 \in Y\}.$$

Theorem

- *Ω_0 is a Cantor set.*
- *T-orbits are path-connected components.*
- *Ω is locally homeomorphic to the product of a Cantor set by $\mathbb{R}^d$*
- *Actually (Ω, T) is a laminated space where the leaves have a flat structure.*

Derived tilings versus box decompositions

Box decompositions:

- A box in Ω is a set $B = C[D] := \{Y - t \mid Y \in C, t \in D\}$ s.t.
  - C is a "local transversal".
  - $D \subset \mathbb{R}^d$ is open.
  - B is homeomorphic to $C \times D$.
- A box decomposition $\mathcal{B} = \{B_1, \ldots, B_n\}$ s.t. pairwise disjoings and their closures cover Ω.

Locally derived tilings:

- A tiling $\mathcal{T}$ is locally derived from a Delone set X if it can be obtained from X by local rules.

Lemma

There is a correspondence between tilings that are locally derivable from X and box decompositions of Ω by the process of unfolding leaves of Ω.

What the heck Does this mean?

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Tower systems

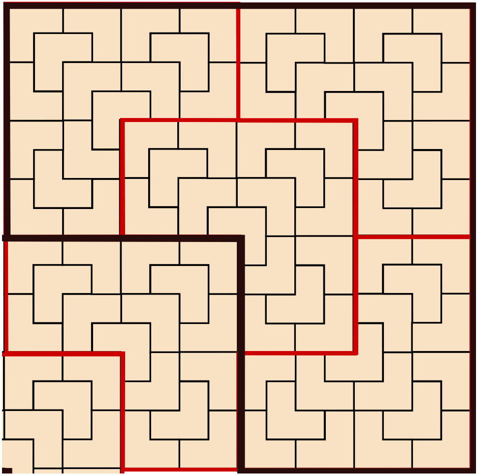
A tower system can be described:

As a sequence $\mathcal{B}_n$ of box decompositions such tht B_{n+1} is zoomed out of B_n. (What is zooming out?)

As a sequence of tilings $\mathcal{T}_n$ such that

- $\mathcal{T}_0$ is locally derivable from X.
- $\mathcal{T}_{n+1}$ is locally derivable from $\mathcal{T}_n$.
- Each tile of $\mathcal{T}_{n+1}$ is a pattern of $\mathcal{T}_n$.

Example: Substitution tilings



Main result

Theorem (A.-P., Coronel)

Let X be an aperiodic linearly repetitive Delone set with constant $L > 1$ and $0 \in X$. Given $K \geq 6L(L+1)^2$ and $s_0 > 0$, set $s_n = K^n s_0$ for all $n \in \mathbb{N}$ and let $C_n := C_{X,s_n}$ for all $n \in \mathbb{N}$. Then, there exists a tower system of Ω adapted to $(C_n)_{n\in\mathbb{N}}$ that satisfies the following additional properties:

(i) *there exist constants $0 < K_1 = K_1(L,K) < 1 < K_2 := K_2(L,K)$ such that for every $n \in \mathbb{N}$ we have*

$$K_1 s_n \leq r_{int}(\mathcal{B}_n) < R_{ext}(\mathcal{B}_n) \leq K_2 s_n; \qquad (4.1)$$

(ii) *for every $n \in \mathbb{N}^*$, the matrix M_n has strictly positive coefficients;*

(iii) *the matrices $\{M_n\}_{n\in\mathbb{N}^*}$ are uniformly bounded in size and norm.*

Remark

Constructive proof. Compare with other results, like Lenz-Stollman 2005. Mention N-Priebe

Applications

Theorem (Lagarias and Pleasants)

Linearly repetitive Delone systems are uniquely ergodic. Moreover, the rate of convergence for frequencies can be estimated.

Theorem (Coronel 2010, Sadun-Frank 2010+)

Linearly repetitive Delone systems are not strongly mixing.

Theorem (Coronel 2010)

The characterization of eigenvalues for minimal cantor systems can be generalized to linearly repetitive Delone systems.

The end

Thanks!!!!!!

A picture