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On the dynamics of non-reducible cylindrical vortices

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Abstract: We study the dynamics of Euclidean isometric extensions of minimal homeomorphisms of compact metric spaces. Under a general hypothesis of homogeneity for the base space, we show that these systems are never minimal, thus extending a classical result of Besicovitch concerning cylindrical cascades. Moreover, using Anosov-Katok type methods, we construct a topologically transitive isometric extension over an irrational rotation with a 2-dimensional fiber.

MSC: 37B05, 37E30, 37F50,

Introduction

This work is a natural companion of [10], where the dynamics of cocycles of isometries of nonpositively curved spaces (including $\mathbb{R}^\ell$) over a minimal dynamics (semigroup action) is studied. While [10] is mostly devoted to the *reducible* case, that is, when there is a continuous invariant section (which turns out to be equivalent to the existence of bounded orbits), here we concentrate on the opposite (*i.e.* non-reducible) case. For simplicity, we restrict our attention to actions of $\mathbb{Z}$. Thus, we consider a minimal homeomorphism $T : X \to X$ from a compact metric space X to itself, and given two continuous functions $\Psi : X \to O(\mathbb{R}^\ell)$ and $\rho : X \to \mathbb{R}^\ell$, we consider the dynamics of the fibered transformation

$$(x, v) \longrightarrow \big(T(x), \Psi(x)v + \rho(x)\big).$$

Such a fibered map will be referred to as a *cylindrical vortex*, a name that is inspired from that of the classical *cylindrical cascades*, which correspond to vortices with $\ell = 1$ and $\Psi(x) = \mathrm{Id}$, for all $x \in X$. One of the main difficulties of our study is that cylindrical vortices do not commute with translations along the fibers. This property holds for cylindrical cascades, and it is actually a fundamental tool for the study of their dynamical properties (*e.g.* the classical proof of Gottschalk-Hedlund's theorem [12]).

Our main result is a generalization of an old result of Besicovitch [6] to our general framework. Because of technical reasons, we restrict ourselves to the case where X is *locally homogeneous*, that is, for every point $x \in X$ and every neighborhood V of x, given $y \in V$ there exists a homeomorphism $h_{V,x,y}$ sending x into y that is the identity outside V. For example, topological manifolds and the Cantor set are locally homogeneous.

Main Theorem. *No cylindrical vortex over a locally homogeneous space is minimal.*

It is worth to stress that the statement above refers to two-side minimality. (All along this work, the word *orbit* for an homeomorphism refers always to a two-side orbit.) Indeed, the version of this result for positive minimality follows from an elementary and classical result of Gottschalk; see [11].

The validity of our Main Theorem in (fiber) dimension 1 is quite natural; for instance, if X is the unit circle, then it follows from and important and difficult theorem of Le Calvez and Yoccoz [26]. However, our proof is much simpler and follows the lines of Besicovitch's, thought it needs a key modification (we have to consider the case where the linear part of the skew dynamics combines Id and −Id.)

In higher dimension, the situation is rather different, and the arguments are of geometric nature. We follow an strategy initiated by Birkhoff [7], strengthened by Pérez-Marco [23] for germs of 2-dimensional homeomorphisms fixing the origin, and adapted by the third-named author for fibered holomorphic maps [24]. Basically, the idea consists in attaching to each cylindrical vortex a totally invariant compact set "at infinity", which allows concluding the non-minimality. The existence of such a compact set is established by an argument of approximation of the base dynamics by periodic ones.

Although non-reducible cylindrical vortices cannot be minimal, they may admit minimal invariant closed subsets. This is for example the case if there are discrete orbits. However, in the case of a (1-dimensional) cascade, the existence of such an orbit gives raise to a nonzero drift. The rest of this work deals with zero-drift cylindrical vortices. After slightly extending a result due to Matsumoto and Shishikura to this setting (*c.f.* Proposition 7), we show how subtle is the higher-dimensional case. As a "concrete" example, we construct a 2-dimensional, topologically transitive cylindrical vortex over an irrational rotation of the circle such that the angle rotation along the fibers is constant and rationally independent of that on the basis. We close this work by discussing the arithmetic properties of the pairs of angles that may appear for such an example. As a straighforward application of the KAM theory, we show that these pairs must satisfy a Liouville type condition provided the corresponding function ρ is smooth.

1 Non minimality of cylindrical vortices

1.1 The 1-dimensional case

In this section we prove that 1-dimensional cylindrical vortices cannot be minimal. To do this, we need to distinguish three cases: when the linear part of the skew dynamics is the identity everywhere, when it coincides with −Id everywhere, and when it combines Id and −Id. The first case was settled by Besicovitch [6]. The second case follows by slightly modifying Besicovitch's arguments. Finally, the third case can be reduced to the second one.

We start by (recalling and) slightly modifying Besicovitch's proof.[1] Consider a cylindrical cascade

$$F\colon (x,v) \mapsto \big(T(x), v+\rho(x)\big),$$

[1]We do this in order to avoid the use of the fact that if F is minimal then, *a-priori*, it must have a dense orbit.

and denote by Π the projection of $X \times \mathbb{R}$ on $\mathbb{R}$. Obviously, if (the Π-projection of) the orbit of a point is bounded either from above or from below, then F cannot be minimal. Assume next that all the orbits are unbounded from above and from below. We will show that, in this case, all the orbits are proper as maps from $\mathbb{Z}$ into $X \times \mathbb{R}$ (compare §2.1). Indeed, if the orbit of a point $(x, v) \in X \times \mathbb{R}$ is (unbounded from above and from below and) not proper, then by examining all possible cases one easily convinces that we may choose three sequences of integers $n_j < n_j' < n_j''$ such that either

$$\begin{gathered}\lim_{j\to\infty} \Pi(F^{n_j}(x,v)) = \lim_{j\to\infty} \Pi(F^{n_j''}(x,v)) \text{ belongs to } [-\infty, +\infty), \\ \Pi(F^{n_j'}(x,v)) > j, \quad \text{and} \quad \Pi(F^{n_j'}(x,v)) = \max_{n_j \le n \le n_j''} \Pi(F^n(x,v)).\end{gathered} \tag{1}$$

or

$$\begin{gathered}\lim_{j\to\infty} \Pi(F^{n_j}(x,v)) = \lim_{j\to\infty} \Pi(F^{n_j''}(x,v)) \text{ belongs to } (-\infty, +\infty], \\ \Pi(F^{n_j'}(x,v)) < -j, \quad \text{and} \quad \Pi(F^{n_j'}(x,v)) = \min_{n_j \le n \le n_j''} \Pi(F^n(x,v)).\end{gathered} \tag{2}$$

Let us first consider the case (1). Set, for every $n_j - n_j' \le n \le n_j'' - n_j'$,

$$z_{j,n} := F^{n_j'+n}\big(x, v - \Pi(F^{n_j'}(x,v))\big).$$

We have:

$$z_{j,0} = \big(T^{n_j'}(x_0), 0\big), \qquad \text{for every } j \in \mathbb{N}, \tag{3}$$

$$z_{j,n} = F^n(z_{j,0}), \quad \Pi(z_{j,n}) \le 0, \qquad \text{for every } n_j - n_j' \le n \le n_j'' - n_j', \quad \text{and} \tag{4}$$

$$n_j - n_j' \to -\infty, \quad n_j'' - n_j' \to +\infty, \qquad \text{as } j \to +\infty. \tag{5}$$

Due to (3), passing to a subsequence, we may assume that $z_{j,0}$ converges to a point $z_0 := (x_0, 0)$. By (4), $z_{j,n}$ converges to $z_n := F^n(z_0)$, for every $n \in \mathbb{Z}$. Finally, using (4) and (5), one may easily see that the orbit $\{z_n\}_{n\in\mathbb{Z}}$ remains bounded from above by zero, which contradicts our assumption. Similarly, in case (2), one may conclude the existence of a point whose orbit remains bounded from below by zero, which again contradicts our assumption.

Consider now a cylindrical vortex of type

$$F\colon (x,v) \mapsto \big(T(x), -v + \rho(x)\big).$$

Set $G := F^2$. This is a fibered transformation over the non-necessarily minimal map T^2. Clearly, if G has a bounded orbit, then the same holds for F, which violates our hypothesis of minimality for F. If not, then the arguments given so far show that either:

– there is a point $z_0^* = (x_0, v_0)$ whose G-orbit is bounded from above, or
– there is a point $z_0^* = (x_0, v_0)$ whose G-orbit is bounded from below, or
– all the G-orbits are proper.

In the first case, we let

$$h := \sup_{n\in\mathbb{Z}} \Pi\big(G^n(x_0, v_0)\big).$$

With obvious notation, for every $w \in \mathbb{R}$, we have

$$\begin{aligned} orb_F(x_0, w) &= orb_G(x_0, w) \bigcup F\big(orb_G(x_0, w)\big) \\ &\subset X \times (-\infty, w - v_0 + h] \bigcup F(X \times (-\infty, w - v_0 + h]) \\ &\subset X \times \big(-\infty, w - v_0 + h + \|\rho\|\big] \bigcup X \times \big[-w + v_0 - h - \|\rho\|, +\infty\big). \end{aligned}$$

Taking $w_0 := -1 + v_0 - h - \|\rho\|$, we see that the F-orbit of (x_0, w_0) avoids $X \times (-1, 1)$. In particular, F is not minimal. The second case can be treated similarly. Finally, since F is a proper map, if the G-orbit of (x, v_0) is proper, then its F-orbit $orb_F(x_0, v_0) = orb_G(x_0, v_0) \cup F\big(orb_G(x_0, v_0)\big)$ is also proper, and F cannot be minimal neither.

Finally, consider a general 1-dimensional cylindrical vortex

$$F : (x, v) \mapsto \big(T(x), \Psi(x)v + \rho(x)\big),$$

where linear part $\Psi(x)$ can be either Id or $-$Id at each point.[2] Denote by $Y \subset X$ the preimage of $\{-\mathrm{Id}\}$ under Ψ. This is a clopen set. Assuming that it is nonempty, the map F_Y induced from F by the first-return map T_Y to the set Y has the form

$$F_Y(x, v) = \big(T_Y(x), -v + \tilde{\rho}(x)\big),$$

where $\tilde{\rho}\colon Y \to \mathbb{R}$. Since T is a minimal homeomorphism, the same must hold for T_Y. Therefore, F_Y is a cylindrical vortex of the type considered in the second case, thus it cannot be minimal. Now, every point in $Y \times \mathbb{R}$ that does not have a dense orbit for F_Y also fails to have a dense orbit under F, since $Y^c \times \mathbb{R}$ is a closed set. Therefore, F is not minimal, and this concludes the proof of the Main Theorem for $\ell = 1$.

1.2 The case of higher dimension

In [7], to each local planar homeomorphism fixing the origin, Birkhoff associated a forward-invariant compact set touching the boundary of the definition domain. In the holomorphic case, this construction was refined in [23] by Pérez-Marco, who constructed a completely invariant compact set touching the boundary. In this section, we construct a Birkhoff/Pérez-Marco like set *at infinity* (a B/P-M set, for short) for cyclindrical vortices of dimension $\ell \geq 2$. To do this, we follow the strategy developed by the third-named author in [24], where he extends the construction of Pérez-Marco to fibered holomorphic maps. Next, we use the B/P-M sets to show the non-minimality of these vortices. It is worth mentioning that this last idea is not completely new, as Besicovitch's work [6] already includes a remark relating forward non-minimality of cylindrical cascades to the existence of Birkhoff invariant sets.

Given $\ell \geq 2$, let $I\colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ be an affine Euclidean isometry, that is,

$$Iv = \Psi v + \rho$$

[2] For a nice discussion of the measure-theoretical cohomological properties of such a map, see [22].

for certain $\Psi \in O(\mathbb{R}^\ell)$ and $\rho \in \mathbb{R}^\ell$. This isometry extends continuously to the one-point (Alexandrov) compactification $\mathbb{R}^\ell \cup \{\infty\}$ by letting $I\infty = \infty$. We will show that the infinity is not an isolated point as an invariant object. More precisely, we will show that given an open, bounded set U, there exists a closed set containing ∞ that is completely invariant under I, touches the boundary ∂U, and is contained in $\mathbb{R}^\ell \setminus U$. We introduce a terminology for this. We say that K is a B/P-M set for I avoiding U if the following conditions hold:

1. $K \subset \mathbb{R}^\ell \setminus U$.
2. $K \cup \{\infty\}$ is compact and connected for the one-point compactification topology of $\mathbb{R}^\ell$.
3. $I(K) = I^{-1}(K) = K$.
4. $K \cap \partial U \neq \emptyset$.

Proposition 1 *For any open, bounded set $U \subset \mathbb{R}^\ell$ and any affine isometry $I : \mathbb{R}^\ell \to \mathbb{R}^\ell$, there exists a B/P-M set K for I avoiding U.*

To prove this proposition, we need an elementary lemma.

Lemma 2 *If the claim of Proposition 1 holds for I^2 and any open, bounded set $U \subset \mathbb{R}^\ell$, then it also holds for I.*

Proof. Set $\mathcal{U} := U \cup I(U)$. This is an open, bounded set. By hypothesis, there exists a B/P-M set $\mathcal{K}$ for I^2 avoiding $\mathcal{U}$. Letting $K := \mathcal{K} \cup I^{-1}(\mathcal{K})$, it is not hard to check that K is a B/P-M set for I avoiding U. □

Proof of Proposition 1. We start with the case $\ell = 2$. Due to the lemma above, we may assume that Ψ preserves orientation, and hence we only need to consider two cases:

- $\Psi = \mathrm{Id}_{\mathbb{R}^2}$: Take two parallel lines in the direction of ρ touching the boundary of the open set U, and such that U lies in between the band determined by these two lines. Then choose K as being the union of the two semi-planes that form the complement of this band. (See Figure 1.)
- Ψ equals the counterclockwise rotation R_α of angle $\alpha \neq 0$: The isometry I fixes the point $v_0 := (\mathrm{Id} - R_\alpha)^{-1}\rho$, and I corresponds to the rotation of angle α centered at this point. Let B be the smallest open ball centered at this point and containing U. We then may choose $K := B^c$. (See Figure 2.)

When $\ell = 3$, in a suitable orthonormal basis, the (orientation-preserving) isometry I may written in the form

$$Iv = \begin{bmatrix} \tilde{\Psi} & \\ & 1 \end{bmatrix} v + \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}$$

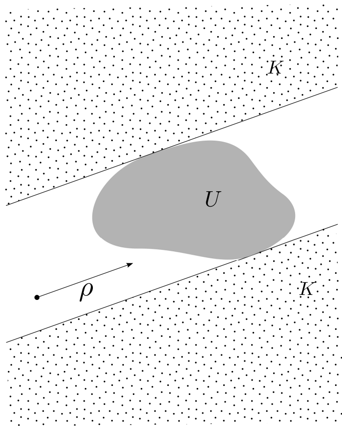

Figure 1

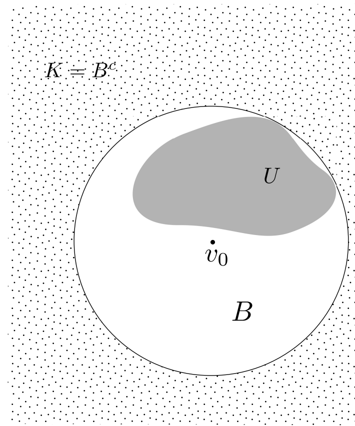

Figure 2

for some linear isometry $\tilde{\Psi}$ of $\mathbb{R}^2$. Let $\tilde{U}$ be the orthogonal projection of U onto $\mathbb{R}^2$, and let $\tilde{K}$ a B/P-M set for $\tilde{I} = \tilde{\Psi} + [a\ b]^T$ avoiding $\tilde{U}$. (Such a set is known to exist because the case $\ell = 2$ has already been settled.) Then $K := \tilde{K} \times \mathbb{R}$ is a B/P-M set avoiding U.

Assume now that the proposition holds for $\ell = k-1$ and $\ell = k$, and consider the case $\ell = k+1$. In a suitable orthonormal basis, the isometry may be written in the form

$$Iv = \begin{bmatrix} \tilde{\Psi} & \\ & \hat{\Psi} \end{bmatrix} v + \begin{bmatrix} a_1 \\ \vdots \\ a_{k-1} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_k \\ a_{k+1} \end{bmatrix}$$

where $\tilde{\Psi}, \hat{\Psi}$ are linear isometries of $\mathbb{R}^{k-1}$ and $\mathbb{R}^2$, respectively. Let $\tilde{U}$ be the orthogonal projection of U onto $\mathbb{R}^{k-1}$, and let $\tilde{K}$ be a B/P-M invariant for $\tilde{I} = \tilde{\Psi} + [a_1 \dots a_{k-1}]^T$ avoiding $\tilde{U}$. Then $K := \tilde{K} \times \mathbb{R}^2$ is a B/P-M set for I avoiding U. □

In what follows, we give an analog of the preceding proposition for fibered isometries. More precisely, we consider a cylindrical vortex $F\colon X \times \mathbb{R}^\ell \to X \times \mathbb{R}^\ell$ over a minimal homeomorphism $T\colon X \to X$. We say that an open, bounded set $U \subset X \times \mathbb{R}^\ell$ is a *tube* for F if for every $x \in X$, the fiber U_x is nonempty. A closed set K is a B/P-M set for F avoiding U if the following conditions hold:

1. $K \subset \big(X \times \mathbb{R}^\ell\big) \setminus U$.

2. For each $x \in X$, the fiber $K_x \cup \{\infty\}$ is compact and connected for the one-point compactification topology of the fiber $\mathbb{R}^\ell$.

3. $F(K) = F^{-1}(K) = K$.

4. There exists $x^* \in X$ such that $K_{x^*} \cap \partial U_{x^*} \neq \emptyset$.

We first consider a fibered dynamics over the finite space $X := \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ with the basis homeomorphism $T(j) = j + 1$. In other words, given a finite family $\{I_0, I_1, \ldots, I_{n-1}\}$ of affine isometries of $\mathbb{R}^\ell$, with $\ell \geq 2$, we consider the cylindrical vortex

$$\begin{array}{rcl} F\colon \mathbb{Z}_p \times \mathbb{R}^\ell & \longrightarrow & \mathbb{Z}_p \times \mathbb{R}^\ell \\ (j, v) & \longmapsto & (j+1, I_j v)\,. \end{array}$$

Proposition 3 *For every tube $U \subset X \times \mathbb{R}^\ell$, there exists a B/P-M invariant set K for F avoiding U.*

Proof. For simplicity, we only deal with the case $p = 2$, leaving the (slightly more elaborate) general case to the reader. Let U_0, U_1 be the two (nonempty) open, bounded subsets of $\mathbb{R}^\ell$ such that $U = (\{0\} \times U_0) \bigcup (\{1\} \times U_1)$, and let $\mathcal{U} := U_0 \cup I_0^{-1}(U_1)$. By Proposition 1, there exists a B/P-M invariant set K_0 for $I_1 \circ I_0$ avoiding $\mathcal{U}$. Set $K_1 := I_0(K_0)$ and $K := (\{0\} \times K_0) \bigcup (\{1\} \times K_1)$. Properties 1. and 2. in the definition above are obvious, whereas Property 3. follows from

$$\begin{array}{rcl} (I_1 \circ I_0)^{-1}(K_0) & = & K_0, \\ I_0^{-1}\left(I_1^{-1}(K_0)\right) & = & K_0, \\ I_1^{-1}(K_0) & = & I_0(K_0) \quad = \quad K_1. \end{array}$$

Since $K_0 \cap \partial \mathcal{U} \neq \emptyset$, either $K_0 \cap \partial U_0 \neq \emptyset$ or $K_0 \cap \partial I_0^{-1}(U_1) \neq \emptyset$. Since the second condition is equivalent to $K_1 \cap \partial U_1 \neq \emptyset$, in both cases Property 4. above holds. $\square$

We can now proceed to the general case covered by the Main Theorem. Slightly more generally, we will say that $T\colon X \to X$ is *approximated by homeomorphisms having periodic orbits* if there exists a sequence of homeomorphisms $T_n\colon X \to X$ that converges to T uniformly on X so that each T_n has a periodic point. Our Main Theorem follows directly from the next two propositions.

Proposition 4 *Let X be a locally homogeneous compact metric space. If $T\colon X \to X$ is a minimal homeomorphism, then T is approximated by homeomorphisms having periodic orbits.*

Proof. Given $x \in X$, let V_n be a decreasing sequence of neighborhoods of x converging to $\{x\}$. Let $p_n \in \mathbb{N}$ be the first-return time of x to V_n under T, and let $y_n := T^{p_n} x$. Define $T_n := h_{V_n, x, y_n} \circ T$. Clearly, x is periodic for T_n with period p_n. Finally, since $V_n \to \{x\}$, we have the (uniform) convergence $T_n \to T$. $\square$

Proposition 5 *Let $U \subset X \times \mathbb{R}^\ell$ be a tube. If $T\colon X \to X$ is approximated by homeomorphisms having periodic orbits, then there exists a B/P-M set for F avoiding U.*

Proof. Let $x_n^0, x_n^1 := T(x_n^0), \ldots, x_n^{p(n)-1} := T(x_n^{p(n)-2})$ be a periodic orbit of period $p(n)$ of T_n, which we identify with $\mathbb{Z}_{p(n)}$. Let $F_n\colon \mathbb{Z}_{p(n)} \times \mathbb{R}^\ell \to \mathbb{Z}_{p(n)} \times \mathbb{R}^\ell$ be the cylindrical vortex defined by

$$F_n(x_n^j, v) = \left(x_n^{j+1}, \Psi(x_n^j) v + \rho(x_n^j)\right).$$

Proposition 3 yields a B/P-M invariant set K_n for F_n avoiding U. Let $\hat{K}_n \subset X \times \overline{\mathbb{R}}^\ell$ be the compact set resulting from K_n by attaching the section $X \times \{\infty\}$ to it, that is, $\hat{K}_n := K_n \cup (X \times \{\infty\})$.

Taking an appropriate subsequence, we may assume that there exists a connected compact set $K \subset X \times \overline{\mathbb{R}}^\ell$ that is the limit of K_n for the Hausdorff topology on compact sets. Since ∂U is a compact set and, for each $n \in \mathbb{N}$, one has $K_n \cap \partial U \neq \emptyset$, the intersection $K \cap \partial U$ is nonempty. We denote $\hat{K} := K \cup (X \times \{\infty\})$. Since $T_n \to T$ uniformly, we have the uniform convergence $F_n \to F$. Thus, $F_n(\hat{K}_n) \to F(\hat{K})$ and $F_n^{-1}(\hat{K}_n) \to F^{-1}(\hat{K})$. This implies that $F(K) = F^{-1}(K) = K$, which closes the proof. □

1.3 Two possible generalizations

Recall that the main theorem of [10] applies not only to cocycles of isometries of a Hilbert space but also to cocycles of isometries of a CAT(0) proper space. It is very likely that our Main Theorem here holds in this context as well. Indeed, there is a general description of isometries of such a space that allows to give an analog of Proposition 1 for "higher dimensional" CAT(0)-spaces (*e.g.* spaces that are not quasi-isometric to the real line). In this situation, this would certainly allow to perform a similar procedure to get a B/P-M set and show the non-minimality, whereas for the "1-dimensional case", it should not be very difficult to adapt the arguments of §1.1. We do not carry out the details of all of this here because we do not see any interesting application except for spaces for which the arguments apply without major modifications (*e.g.* hyperbolic spaces $\mathbb{H}^n$). In this direction, it is worth pointing out that, though no cylindrical vortex of isometries of $\mathbb{H}^2$ over an irrational rotation is minimal, the action on the product of the circle and the boundary of the Poincaré disk appears to be minimal in many cases [4].

Perhaps more interesting is trying to settle the infinite dimensional case of the Main Theorem. Indeed, most of the arguments of §1.2 strongly depend on the fact that the fiber space, namely $\mathbb{R}^\ell$, is locally compact (the same would apply to proper CAT(0)-spaces). This turns natural the following

Question. Does there exist a *minimal* cylindrical vortex with infinite-dimensional fiber ?

Recall that, by [21, Exercise 5.3.15], no isometry of a Hilbert space can be minimal (it cannot be even topologically transitive "at large scale"). However, isometries without fixed points but having recurrent points do exist: see [9] and [21, Exercise 5.2.26]. The situation might be compared with that of general linear maps: in finite dimension, topological transitivity is impossible, while in infinite dimension, even topological mixing may hold [5].

2 Minimal invariant sets

2.1 Almost-reducibility and proper orbits

The next proposition is folklore but difficult to find in the literature stated in this way (compare [16, 20]); we include the proof just for the convenience of the reader. For the statement, notice that for a cylindrical cascade $F\colon (x, v) \to \big(T(x), v + \rho(x)\big)$ and each $n \in \mathbb{N}$,

$$F^n(x, v) = \big(T^n(x), v + \rho_n(x)\big),$$

where ρ_n denotes the Birkhoff sum

$$\rho_n(x) := \sum_{j=0}^{n-1} \rho\big(T^j(x)\big).$$

Proposition 6 *The following properties are equivalent:*

1. *There exists a family of continuous sections $\varphi_k \colon X \to \mathbb{R}$ that is almost invariant under the skew action. In other words, the associate cohomological equation can be solved in reduced cohomology:*
$$\lim_{k\to+\infty} \left(\sup_{x\in X} \left|\rho(x) - \big[\varphi_k(T(x)) - \varphi_k(x)\big]\right| \right) = 0.$$

2. *We have the convergence*
$$\lim_{n\to+\infty} \left(\sup_{x\in X} \left|\frac{\rho_n(x)}{n}\right| \right) = 0.$$

3. *The union of proper orbits has zero μ-measure for every T-invariant probability measure μ.*

4. *For every T-invariant (ergodic) probability measure μ on X,*
$$\int_X \rho(x) d\mu(x) = 0.$$

Proof. We give the proof of more implications than necessary because they will useful for the further discussion of general cylindrical vortices.
1. $\to$ *2.* Given $\varepsilon > 0$, let $k = k(\varepsilon)$ be such that for all $x \in X$,

$$\big|\rho(x) - [\varphi_k(T(x)) - \varphi_k(x)\big| \leq \varepsilon.$$

For each $n \in \mathbb{N}$,

$$\big|\rho_n(x) - [\varphi_k(T^n(x)) - \varphi_k(x)]\big| \leq n\varepsilon,$$

thus

$$\Big|\frac{\rho_n(x)}{n}\Big| \leq \frac{2\|\varphi_k\|_\infty}{n} + \varepsilon.$$

Passing to the limit, this yields, with an uniform rate,

$$\lim_{n\to+\infty} \Big|\frac{\rho_n(x)}{n}\Big| \leq \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we have the desired uniform convergence to zero.
2. $\to$ *1.* For each $k \in \mathbb{N}$, let

$$\varphi_k(x) := -\frac{\rho_1(x) + \rho_2(x) + \cdots + \rho_k(x)}{k}.$$

We have

$$\varphi_k(T(x)) - \varphi_k(x) = \frac{1}{k}\sum_{j=1}^{k}\big[\rho_j(x) - \rho_j(T(x))\big] = \frac{1}{k}\sum_{j=1}^{k}\big[\rho(x) - \rho(T^j(x))\big] = \rho(x) - \frac{\rho_k(T(x))}{k},$$

and the last expression converges uniformly to $\rho(x)$.

1. $\to$ *4.* For every T-invariant probability measure μ and each $k \in \mathbb{N}$, we have

$$\int_X \big[\varphi_k(T(x)) - \varphi(x)\big]\, d\mu = 0.$$

Thus,

$$\int_X \rho\, d\mu = \int_X \lim_k \big[\varphi_k \circ T - \varphi\big]\, d\mu = \lim_k \int_X \big[\varphi_k(T(x)) - \varphi(x)\big]\, d\mu = 0.$$

4. $\to$ *1.* Let $\mathbb{L}$ be the closure of the subspace of $C(X)$ spanned by the functions of the form $\varphi \circ T - \varphi$, where $\varphi \in C(X)$. If ρ does not belong to $\mathbb{L}$, the Hahn-Banach separation theorem provides us with a linear functional I that restricted to $\mathbb{L}$ is zero and $I(\rho) = 1$. Such an I comes from integration with respect to a signed probability measure on X. Since $I(\mathbb{L}) = \{0\}$, this measure is T-invariant. Finally, the Hahn decomposition theorem yields an invariant probability measure for which the integral of ρ is nonzero.

2. $\to$ *3.* This implication directly follows from a classical lemma due to Kesten [14] (it can be also derived from a well-known lemma of Atkinson [2]).

3. $\to$ *4.* This follows directly from the Birkhoff ergodic theorem. □

Notice that F is reducible if and only if the cohomological equation

$$\rho(x) = \varphi(T(x)) - \varphi(x)$$

has a continuous solution $\varphi\colon X \to \mathbb{R}$. We will hence say that the cylindrical cascade F is *almost-reducible* if the equivalent conditions of the preceding proposition hold.

The situation for a cylindrical vortex

$$F\colon (x, v) \to \big(T(x), \Psi(x)v + \rho(x)\big)$$

is less transparent. First, in order to introduce a drift-like condition, notice that if we define $\rho_n\colon X \to \mathbb{R}$ (and $\Psi_n\colon X \to O(\mathbb{R}^\ell)$) by

$$F^n(x, v) =\colon \big(T^n(x), \Psi_n(x)v + \rho_n(x)\big),$$

then for all m, n in $\mathbb{Z}$, we have

$$\rho_{m+n}(x) = \Psi_n(T^m(x))\big(\rho_m(x)\big) + \rho_n(T^m(x)).$$

In particular,

$$\big\|\rho_{m+n}(x)\big\| \leq \big\|\rho_m(x)\big\| + \big\|\rho_n(T^m(x))\big\|.$$

By the sub-additive ergodic theorem [15], for every T-invariant ergodic probability measure μ, the value of

$$\frac{\|\rho_n(x)\|}{n}$$

converges to a limit (drift) $D = D(\mu)$ for μ-almost every point $x \in X$.

The main difference here is that Proposition 6 does not extend to cylindrical vortices, even to those with $\Psi \equiv \mathrm{Id}$. More precisely, the equivalence between conditions 1. and 2. still holds with an analogous (direct) proof. (See [8] for a more general version of this fact.) Nevertheless, all the arguments relying on ergodic type theorems fail. As a matter of example, let us consider Yoccoz' example from [26]. This is an irrational rotation of the 2-torus $T\colon (x,y) \mapsto (x+\alpha, y+\beta)$ together with two continuous functions $\hat{\rho} = \hat{\rho}(x)$ and $\check{\rho} = \check{\rho}(y)$, both of zero integral, such that for almost every $(x,y) \in \mathbb{T}^2$,

$$\lim_{n\to\pm\infty} \big[|\hat{\rho}_n(x)| + |\check{\rho}_n(y)|\big] = \infty.$$

Then letting $\Psi \equiv \mathrm{Id}$ and $\rho := (\hat{\rho}, \check{\rho})$, almost every orbit of the induced cylindrical vortex F on $\mathbb{T}^2 \times \mathbb{R}^2$ is proper. However, for *every* point $(x,y) \in \mathbb{T}^2$, we have

$$\limsup_{n\to+\infty} \frac{\|\rho_n(x,y)\|}{n} = \limsup_{n\to+\infty} \frac{|\hat{\rho}_n(x)| + |\check{\rho}_n(y)|}{n} = 0,$$

where the convergence is uniform in (x,y). In particular, $D = D(\mathrm{Leb}) = 0$.

2.2 On a theorem of Matsumoto and Shishikura

One of the major interests on proper orbits is that they are nontrivial minimal invariant closed sets. It easily follows from Denjoy-Koksma's inequality that for any almost-reducible cylindrical cascade over an irrational circle rotation, there is no such orbit provided that the function ρ has finite total variation (without the last assumption, proper orbits may appear; see [25] and [6]; see also [17] for recent simpler examples). Actually, as it is shown by Matsumoto and Shishikura in [19], no nonempty, proper, minimal invariant closed set can appear in this situation. A slight extension of this result is our next

Proposition 7 *Let F be an almost-reducible, 1-dimensional cylindrical vortex over an irrational rotation of the circle. If the corresponding function ρ has finite total variation, then F admits no nonempty, proper, minimal invariant closed set.*

Proof. Let $F : (x,v) \to \big(x+\alpha, -v+\rho(x)\big)$ be a cylindrical vortex satisfying the hypothesis. (If $\Psi \equiv \mathrm{Id}$, then Matsumoto-Shishikura's result applies.) Since F is assumed to be almost-reducible, the same must hold for F^2. Notice that F^2 is a cylindrical cascade to which Matsumoto-Shishikura's theorem applies.

Let $C \neq \emptyset$ be a nonempty minimal closed F-invariant subset of $\mathbb{T}^1 \times \mathbb{R}$. Given $(x,v) \in C$, denote by $C_{(x,v)}$ the closure of its (full) orbit under F^2. We will show below that either C is F^2-minimal or $C_{(x_0,v_0)}$ is F^2-minimal for some (x_0,v_0). Before showing this, notice that the previous remark yields either $C = X \times \mathbb{R}$ or $C_{(x_0,v_0)} = X \times \mathbb{R}$, respectively. Since $C_{(x_0,v_0)} \subset C$, in the second case we still have $C = \mathbb{T}^1 \times \mathbb{R}$, as we wanted to check.

Assume that no $C_{(x,v)}$ is F^2-minimal, and let $C^*_{(x,v)} \subsetneq C_{(x,v)}$ be a nonempty closed F^2-invariant set. Then the set $C^*_{(x,v)} \cup F(C^*_{(x,v)})$ is nonempty, closed, F-invariant and contained in C. By minimality, it coincides with C. Now choose (y,w) (depending on (x,v)) in $C_{(x,v)} \setminus C^*_{(x,v)}$. We must have $(y,w) \in F(C^*_{(x,v)}) \subset F(C_{(x,v)})$. Thus, the closed set $C_{(x,v)} \cap F(C_{(x,v)})$ is nonempty. Since it is F-invariant, it must coincide with C, and by minimality, this easily implies that $C_{(x,v)} = C$. Finally, since this holds for every $(x,v) \in C$, this shows that C is F^2-minimal, which concludes the proof. □

2.3 An interesting family of cylindrical vortices

Following [10, Example 3], given two rationally independent angles α, β and a continuous function $\rho : \mathbb{T}^1 \to \mathbb{C}$, we consider the cylindrical vortex $F : (x,z) \mapsto \big(x+\alpha, e^{2\pi i\beta} z + \rho(x)\big)$ on $\mathbb{T}^1 \times \mathbb{C}$.

Lemma 8 *The map F has zero drift and is conservative (that is, it admits no wandering open domain).*

Proof. The function ρ_n defined so that

$$F^n(x,z) = \big(x + n\alpha, e^{2\pi i n\beta} z + \rho_n(x)\big) \tag{6}$$

may be rewritten in the form

$$\rho_n(x) = \sum_{k=0}^{n-1} e^{2\pi i(n-k-1)\beta} \rho(x+k\alpha) = e^{2\pi i n\beta} \sum_{k=0}^{n-1} e^{-2\pi i(k+1)\beta} \rho(x+k\alpha).$$

Up to the factor $e^{2\pi i n\beta}$, this coincides with the n^{th} Birkhoff sum $S_n^T(\chi)(x,0)$ at the point $(x,0) \in \mathbb{T}^2$ of the function $\chi(x,y) = e^{2\pi i(y-\beta)}\rho(x)$ with respect to the dynamics of the rotation $T\colon \mathbb{T}^2 \to \mathbb{T}^2$ of angle $(\alpha, -\beta)$. Indeed,

$$S_n^T(\chi)(x,y) = \sum_{k=0}^{n-1} e^{2\pi i(y-(k+1)\beta)} \rho(x+k\alpha), \qquad \rho_n(x) = e^{2\pi i n\beta} S_n^T(\chi)(x,0). \tag{7}$$

In particular,

$$\left|\frac{\rho_n(x)}{n}\right| = \left|\frac{S_n^T(\chi)(x,0)}{n}\right|. \tag{8}$$

Since α, β are rationally independent, the map T is uniquely ergodic. Since χ is continuous, the right side member of (8) uniformly converges to

$$\int_{\mathbb{T}^2} \chi(x,y)\, dx\, dy = \int_{\mathbb{T}^1}\int_{\mathbb{T}^1} e^{2\pi i(y-\beta)} \rho(x)\, dx\, dy = \int_{\mathbb{T}^1} e^{2\pi i(y-\beta)} dy \int_{\mathbb{T}^1} \rho(x) dx = 0.$$

In particular, the drift of F is zero.

To show the conservativity of F notice that, due to Atkinson's lemma [2], if we fix $x \in \mathbb{T}^1$ and $\varepsilon > 0$, there exist $\bar{x} \in \mathbb{T}^1$, $y \in \mathbb{T}^1$, and $n \in \mathbb{N}$, such that $S_n^T(\chi)(\bar{x}, y) \leq \varepsilon$ and

$$\operatorname{dist}(x, \bar{x}) \leq \varepsilon, \quad \operatorname{dist}(\bar{x}, \bar{x} + n\alpha) \leq \varepsilon, \quad \operatorname{dist}(y, 0) \leq \varepsilon, \quad \operatorname{dist}(y, y - n\beta) \leq \varepsilon.$$

This implies that both $n\alpha$ and $n\beta$ are ε-close to zero. Together with (6), (7), and

$$\operatorname{dist}(x, \bar{x}) \leq \varepsilon, \qquad S_n^T(\chi)(\bar{x}, y) \leq \varepsilon,$$

this implies that, for any $z \in \mathbb{C}$, the n^{th}-iterate under F of the $(\varepsilon, \varepsilon)$-neighborhood of (x, z) is $(\varepsilon, \varepsilon(\|z\| + 1))$-close to it. Therefore, F has no wandering open domain. $\square$

As shown by the proof above, the dynamics of F is closely related to the cylindrical cascade G on $\mathbb{T}^2 \times \mathbb{C}$ defined by

$$G\big((x, y), z\big) \mapsto \big((x + \alpha, y - \beta), z + e^{2\pi i(y-\beta)} \rho(x)\big).$$

Besides (7), both maps are related in that F is a factor of G. Indeed, letting $\Pi \colon \mathbb{T}^2 \times \mathbb{C} \to \mathbb{T}^1 \times \mathbb{C}$ be the *proper map* defined by

$$\Pi\big((x, y), z\big) := (x, e^{-2\pi i y} z),$$

we have

$$F \circ \Pi = \Pi \circ G.$$

These relations allow showing the next

Lemma 9 *The map F is reducible if and only if G is.*

Proof. In view of either the second relation in (7) or the fact that F is a factor of G by a proper map, this follows as a direct application of the main result of [10]. A direct argument proceeds as follows.

Recall that for G being reducible we mean that there exists a continuous function $\varphi \colon \mathbb{T}^2 \to \mathbb{C}$ such that, for all $(x, y) \in \mathbb{T}^2$,

$$\varphi(x + \alpha, y - \beta) = \varphi(x, y) + e^{2\pi i(y-\beta)} \rho(x).$$

If this holds, then defining the continuous function $\varphi_* \colon \mathbb{T}^1 \to \mathbb{C}$ by

$$\varphi_*(x) := \int_{\mathbb{T}^1} e^{-2\pi i y} \varphi(x, y) dy,$$

we obtain

$$\begin{aligned} \varphi_*(x) &= \int_{\mathbb{T}^1} \big[e^{-2\pi i y} \varphi(x + \alpha, y - \beta) - e^{-2\pi i \beta} \rho(x)\big] dy \\ &= e^{-2\pi i \beta} \int_{\mathbb{T}^1} e^{-2\pi i(y-\beta)} \varphi(x + \alpha, y - \beta) dy \; - \; e^{-2\pi i \beta} \rho(x) = e^{-2\pi i \beta} \big[\varphi_*(x + \alpha) - \rho(x)\big]. \end{aligned}$$

Hence

$$\varphi_*(x+\alpha) = e^{2\pi i\beta}\varphi_*(x) + \rho(x),$$

which shows that F is reducible.

Conversely, assume that F is reducible, that is, there exists a continuous function $\varphi_*\colon \mathbb{T}^1 \to \mathbb{C}$ such that

$$\varphi_*(x+\alpha) = e^{2\pi i\beta}\varphi_*(x) + \rho(x).$$

If we multiply by $e^{2\pi i(y-\beta)}$ both sides of this equality, we obtain

$$e^{2\pi i(y-\beta)}\varphi_*(x+\alpha) = e^{2\pi iy}\varphi_*(x) + e^{2\pi i(y-\beta)}\rho(x).$$

Therefore, if we define $\varphi\colon \mathbb{T}^2 \to \mathbb{C}$ by $\varphi(x,y) := e^{2\pi iy}\varphi_*(x)$, we have

$$\varphi(x+\alpha, y-\beta) = \varphi(x,y) + e^{2\pi i(y-\beta)}\rho(x),$$

thus showing that G is reducible. □

A more elaborate relation between F and G is given by the next

Proposition 10 *The map F is topologically transitive if and only if G is.*

To show (the difficult implication of) this proposition, we will strongly use a deep theorem due to Atkinson that characterizes the failure of topological transitivity by the existence of reducible linear factors [3]. In our case, this may be stated as follows:

Theorem 11 [Atkinson] *Assuming that G is conservative and non-reducible, a necessary and sufficient condition for its topological transitivity is that there is no $\theta \in \mathbb{T}^1$ such that the 1-dimensional cylindrical cascade*

$$\big((x,y),t\big) \mapsto \big((x+\alpha, y-\beta), t + \langle e^{2\pi i\theta}, e^{2\pi i(y-\beta)}\rho(x)\rangle\big) \tag{9}$$

is reducible, where $\langle\cdot,\cdot\rangle$ stands for the inner product of vectors in $\mathbb{R}^2 \sim \mathbb{C}$.

Proof of Proposition 10. Since F is a factor of G, the map F is topologically transitive whenever G is.

To prove the converse implication, assume first that (9) is reducible for some $\theta \in \mathbb{T}^1$, that is, there exists a continuous function $\varphi\colon \mathbb{T}^2 \to \mathbb{R}$ such that, for all $(x,y) \in \mathbb{T}^2$,

$$\langle e^{2\pi i\theta}, e^{2\pi i(y-\beta)}\rho(x)\rangle = \varphi(x+\alpha, y-\beta) - \varphi(x,y).$$

Then, letting

$$\varphi_\vartheta(x,y) := \varphi(x, y-\vartheta),$$

we have

$$\langle e^{2\pi i\theta'}, e^{2\pi i(y-\beta)}\rho(x)\rangle = \varphi_{\theta'-\theta}(x+\alpha, y-\beta) - \varphi_{\theta'-\theta}(x,y).$$

In particular, both cocycles $\mathrm{Re}(e^{2\pi i(y-\beta)}\rho(x)) = \langle 1, e^{2\pi i(y-\beta)}\rho(x)\rangle$ and $\mathrm{Im}(e^{2\pi i(y-\beta)}\rho(x)) = \langle i, e^{2\pi i(y-\beta)}\rho(x)\rangle$ are reducible. Hence, the cocycle $e^{2\pi i(y-\beta)}\rho(x)$ is reducible, which is impossible since G is topologically transitive.

Now, if no cylindrical cascade (9) is reducible, then in order to apply Atkinson's theorem for concluding that G is topologically transitive, we need to show that G is conservative whenever F is topologically transitive. To do this, we first notice that the non-wandering set of G is invariant under both G and the translations along the fibers

$$\big((x,y),z\big) \mapsto \big((x,y),z+t\big), \quad t \in \mathbb{C}.$$

Therefore, this set is either empty or the whole space $\mathbb{T}^2 \times \mathbb{C}$. To exhibit a non-wandering point of G we proceed as follows. Since F is topologically transitive, we may chose a point (x_0, z_0) in $\mathbb{T}^1 \times \mathbb{C}$ having a dense orbit under F. If we denote $(x_n, z_n) := F^n(x_0, z_0)$, this implies in particular that there exists a strictly monotone sequence of integers (n_k) such that $(x_{n_k}, z_{n_k}) \to (x_0, z_0)$ as $k \to \infty$. Since Π is a proper map, the sequence of subsets $\Pi^{-1}(x_{n_k}, z_{n_k}) \subset \mathbb{T}^2 \times \mathbb{C}$ remains inside a compact set. In particular, there must be a point $\big((\tilde{x}_0, \tilde{y}_0), \tilde{z}_0\big) \in \Pi^{-1}(x_0, z_0)$ such that the sequence $\big((\tilde{x}_{n_k}, \tilde{y}_{n_k}), \tilde{z}_{n_k}\big) := G^{n_k}\big((\tilde{x}_0, \tilde{y}_0), \tilde{z}_0\big)$ accumulates at some point $\big((\tilde{x}_\infty, \tilde{y}_\infty), \tilde{z}_\infty\big)$. As it is easy to check, every such an accumulation point is non-wandering for G. $\square$

The construction of a map

$$F\colon (x,z) \mapsto \big(x+\alpha, e^{2\pi i\beta} z + \rho(x)\big)$$

that is topologically transitive will be the main issue of the next section. Let us close this section by pointing out that we do not know whether there exists a cylindrical vortex F of the form above that is neither reducible nor topologically transitive. This is related to the existence of Yoccoz-like examples (see [26]) associated to functions of a particular form, which seems to be a difficult problem. Indeed, the previous arguments easily show the following

Proposition 12 *If F is neither reducible nor topologically transitive, then G is non-conservative. In particular, the function $(x,y) \mapsto e^{-2\pi i(y-\beta)}\rho(x)$ does not satisfy the Denjoy-Koksma property.*

2.4 A "concrete" example

Quite surprisingly, to perform our construction we will need a lemma about the density of a certain set obtained by an arithmetic type construction.[3] Given $q \in \mathbb{N}$, let $t(q) := [\sqrt[3]{q}]$ and $r(q) := [\sqrt{q}]$. If q and $t(q)$ are coprime, define $p(q) \in \{1, \ldots, q-1\}$ as the inverse (mod q) of $t(q)$. Otherwise, set $p(q) := 0$. Now, let

$$\mathrm{FS}_q(2,3) := \left\{ \Big(\frac{sp(q)}{q}, \frac{sp(q)r(q)}{q} \Big), \ 1 \leq s \leq 2t(q) \right\},$$

[3] We strongly believe that a much more general result should be true; in particular the set $\mathrm{FS}(m,n)$ analogous to that defined further one should be dense for all $m > n$. Nevertheless, we were unable to produce a conceptual proof of this seemingly interesting fact.

where each coordinate is reduced modulo $\mathbb{Z}$ (thus, $\mathrm{FS}_q(2,3)$ is though of as a subset of $[0,1]^2$). Finally, let us consider the *set of frequencies*

$$\mathrm{FS}(2,3) := \bigcup_{q \in \mathbb{N}} \mathrm{FS}_q(2,3).$$

Lemma 13 *The set* $\mathrm{FS}(2,3)$ *is dense in* $[0,1]^2$.

Proof. For a fixed $m \in \mathbb{N}$, let $q := m^6 + 2m^4 + m^2 + 1 = (m^3+m)^2 + 1$. One readily checks that $t(q) = m^2$ and $r(q) = m^3 + m$. Since $t(q)(m^4 + 2m^2 + 1) = m^6 + 2m^4 + m^2 = q - 1$, we have

$$t(q)(q - m^4 - 2m^2 - 1) \equiv 1 \pmod{q}.$$

Hence,

$$p(q) = q - m^4 - 2m^2 - 1 = m^6 + m^4 - m^2.$$

In particular, modulo $\mathbb{Z}$,

$$\frac{p(q)}{q} = -\frac{m^4 + 2m^2 + 1}{m^6 + 2m^4 + m^2 + 1}.$$

Similarly, we have the equality

$$\frac{p(q)r(q)}{q} = -\frac{(m^4+2m^2+1)(m^3+m)}{m^6+2m^4+m^2+1} = -\frac{m^7+3m^5+3m^3+m}{m^6+2m^4+m^2+1} = -m - \frac{m^5+2m^3}{m^6+2m^4+m^2+1}.$$

Hence, modulo $\mathbb{Z}$,

$$\Big(\frac{sp(q)}{q}, \frac{sp(q)r(q)}{q}\Big) = \Big(- \frac{s(m^4+2m^2+1)}{m^6+2m^4+m^2+1}, -\frac{s(m^5+2m^3)}{m^6+2m^4+m^2+1}\Big).$$

Let us consider all possible s of the form $s(j,k) := jm + k$, where j,k range from 1 to m. Notice that all these values satisfy the restriction $1 \leq s \leq 2t(q) = 2m^2$, and therefore the associated pairs

$$\Big(- \frac{(jm+k)(m^4+2m^2+1)}{m^6+2m^4+m^2+1}, -\frac{(jm+k)(m^5+2m^3)}{m^6+2m^4+m^2+1}\Big)$$

belong to $\mathrm{FS}(2,3)$. Now, easy computations show that the pair above coincides with

$$\Big(- \frac{jm^5 + km^4 + 2jm^3 + 2km^2 + jm + k}{m^6 + f_4(m)}, -j - \frac{km^5 + 2km^3 - jm^2 - j}{m^6 + f_4(m)}\Big)$$

where f_4 is a degree-4 polynomial. Since both j,k are (positive and) smaller than or equal to m, for m large-enough, the pair above is very close (modulo $\mathbb{Z}$) to $(-\frac{j}{m}, -\frac{k}{m}) \sim (1 - \frac{j}{m}, 1 - \frac{k}{m})$ with an error that converges to zero as $m \to \infty$ (independently of j,k). As a consequence, every pair of rational numbers in $[0,1]^2$ is contained in the closure of $\mathrm{FS}(2,3)$, which shows that this set is dense. $\square$

To construct our desired topologically transitive cylindrical vortex

$$F\colon (x,z) \mapsto \big(x+\alpha, e^{2\pi i\beta} z + \rho(x)\big),$$

we will perform a sequence of approximations inspired in the classical Anosov-Katok's method [1]. More precisely, we will construct a sequence of skew maps

$$F_k : (x,z) \mapsto \big(x+\alpha_k, e^{2\pi i\beta_k} z + \rho_k(x)\big), \quad \alpha_k \in \mathbb{Q}, \ \ \beta_k \in \mathbb{Q},$$

over periodic rotations so that they converge uniformly on compact sets. The main point consists in prescribing a sequence of sections $\varphi_k\colon \mathbb{T}^1 \to \mathbb{C}$ whose images become more and more dense in larger and larger regions of $\mathbb{C}$ and that are invariant under F_k, that is,

$$\rho_k(x) = \varphi_k(x+\alpha_k) - e^{2\pi i\beta_k}\varphi_k(x).$$

The construction of the sequence (φ_k) is made so that $\varphi_k := \varphi_{k-1} + \psi_k$, where ψ_k has the form

$$\psi_k(x) := \ell_k \sin(2\pi t_k x) e^{2\pi i r_k x}.$$

To perform this construction, we will need to define inductively the sequences (ℓ_k), (r_k), (t_k), (α_k) and (β_k). The choice is irrelevant for $k=1$ (we just need to impose the condition $\ell_1 \geq 1$). Now, assuming that these values have been already defined for $k \in \mathbb{N}$, we let $C_k \geq 1$ be a Lipschitz constant for φ_k and $D_k \geq 1$ be the supremum of the norm of φ_k. According to Lemma 13, we may choose $(\alpha_{k+1}, \beta_{k+1}) := (\frac{s_{k+1} p_{k+1}}{q_{k+1}}, r_{k+1}\alpha_{k+1})$, where p_{k+1} is the inverse (modulo q_{k+1}) of $t_{k+1} := [\sqrt[3]{q_{k+1}}]$ and $r_{k+1} := [\sqrt{q_{k+1}}]$, in such a way that the following conditions are satisfied:

1. $1 \leq s_{k+1} \leq 2t_{k+1}$.
2. $q_{k+1} > q_k^3$, $\ q_{k+1}^{5/12} \geq 2^{k+1} q_k$, $\ q_{k+1}^{1/12} \geq 2k q_k^{1/12}$ and $q_{k+1} \geq \big[2(k+1)\sum_{j=1}^k \ell_j \sqrt{q_j}\big]^3$.
3. $\big|\alpha_{k+1} - \alpha_k\big| \leq \frac{1}{2^{k+2} q_k C_k}$.
4. $\big|\beta_{k+1} - \beta_k\big| \leq \frac{1}{2^{k+2} q_k D_k}$.

Finally, we let $\ell_{k+1} := q_{k+1}^{1/12}$.

For each k, n in $\mathbb{N}$, we define $(\rho_k)_n \colon \mathbb{T}^1 \to \mathbb{C}$ by

$$(\rho_k)_n(x) := \varphi_k(x+n\alpha_k) - e^{2\pi i n\beta_k}\varphi_k(x).$$

Then we have

$$F_k^n(x,z) = \big(x+n\alpha_k, e^{2\pi i n\beta_k} z + (\rho_k)_n(x)\big).$$

The next lemma yields a quantitative estimate for the convergence of the maps F_k as well as some of their iterates.

Lemma 14 *For each $k \in \mathbb{N}$ and $1 \leq n \leq q_k$,*

$$\big|(\rho_{k+1})_n - (\rho_k)_n\big| \leq \frac{\ell_{k+1} q_k s_{k+1}}{q_{k+1}} + C_k q_k |\alpha_{k+1} - \alpha_k| + D_k q_k |\beta_{k+1} - \beta_k|. \tag{10}$$

Proof. Notice that $(\rho_{k+1})_n(x) - (\rho_k)_n(x)$ equals

$$\begin{aligned}
&\varphi_{k+1}(x+n\alpha_{k+1}) - e^{2\pi in\beta_{k+1}}\varphi_{k+1}(x) - \big[\varphi_k(x+n\alpha_k) - e^{2\pi in\beta_k}\varphi_k(x)\big] \\
&\quad= \varphi_{k+1}(x+n\alpha_{k+1}) - \varphi_k(x+n\alpha_k) + e^{2\pi in\beta_k}\varphi_k(x) - e^{2\pi in\beta_{k+1}}[\varphi_k(x)+\psi_{k+1}(x)] \\
&\quad= \big[\psi_{k+1}(x+n\alpha_{k+1}) - e^{2\pi in\beta_{k+1}}\psi_{k+1}(x)\big] + \big[\varphi_k(x+n\alpha_{k+1}) - \varphi_k(x+n\alpha_k)\big] + \varphi_k(x)\big[e^{2\pi in\beta_k} - e^{2\pi in\beta_{k+1}}\big].
\end{aligned}$$

Thus, the value of $\big|(\rho_{k+1})_n(x) - (\rho_k)_n(x)\big|$ is smaller than or equal to

$$\ell_{k+1}\big|\sin(2\pi t_{k+1}(x+n\alpha_{k+1}))e^{2\pi ir_{k+1}(x+n\alpha_{k+1})} - e^{2\pi in\beta_{k+1}}\sin(2\pi t_{k+1}x)e^{2\pi ir_{k+1}x}\big| + C_k n|\alpha_{k+1}-\alpha_k| + D_k n|\beta_{k+1}-\beta_k|.$$

The first term of this expression is bounded from above by

$$\begin{aligned}
&\big|\ell_{k+1}e^{2\pi ir_{k+1}(x+n\alpha_{k+1})}\big[\sin(2\pi t_{k+1}(x+n\alpha_{k+1})) - \sin(2\pi t_{k+1}x)\big] + \ell_{k+1}\sin(2\pi t_{k+1}x)\big[e^{2\pi ir_{k+1}(x+n\alpha_{k+1})} - e^{2\pi i(n\beta_{k+1}+r_{k+1}x)}\big]\big| \\
&\quad\leq \ell_{k+1}\big(\{t_{k+1}n\alpha_{k+1}\} + |e^{2\pi inr_{k+1}\alpha_{k+1}} - e^{2\pi in\beta_{k+1}}|\big).
\end{aligned}$$

Since $p_{k+1}t_{k+1} \equiv 1 \pmod{q_{k+1}}$, we have

$$\{t_{k+1}n\alpha_{k+1}\} = \left\{\frac{ns_{k+1}t_{k+1}p_{k+1}}{q_{k+1}}\right\} = \left\{\frac{ns_{k+1}}{q_{k+1}}\right\}.$$

Moreover, since $\beta_{k+1} = r_{k+1}\alpha_{k+1}$, the term $|e^{2\pi inr_{k+1}\alpha_{k+1}} - e^{2\pi in\beta_{k+1}}|$ vanishes. We thus obtain

$$\big|(\rho_{k+1})_n(x) - (\rho_k)_n(x)\big| \leq \frac{\ell_{k+1}ns_{k+1}}{q_{k+1}} + C_k n|\alpha_{k+1}-\alpha_k| + D_k n|\beta_{k+1}-\beta_k|,$$

which shows the lemma. □

The next lemma deals with the density of the invariant curve $x \mapsto (x, \varphi_k(x))$ inside a large region of $\mathbb{T}^1 \times \mathbb{C}$.

Lemma 15 *For every $k \geq 2$, the graph of φ_k is $(\frac{1}{t_k}, \frac{\ell_k t_k}{r_k} + \frac{4\pi}{t_k}\sum_{j=1}^{k-1}\ell_j r_j)$-dense in the cylinder $\mathbb{T}^1 \times \mathrm{Ball}\big(0, \ell_k - \sum_{j=1}^{k-1}\ell_j\big)$.*

Proof. We first claim that the graph of ψ_k is $(\frac{1}{t_k}, \frac{\ell_k t_k}{r_k})$-dense in $\mathbb{T}^1 \times \mathrm{Ball}(0, \ell_k)$. Indeed, given (x, z) in this cylinder, there must exist $\tilde{x} \in \big[x - \frac{1}{2t_k}, x + \frac{1}{2t_k}\big]$ such that $\psi_k(\tilde{x}) = |z|$. The claim then follows by noticing that $\frac{1}{r_k} \leq \frac{1}{t_k}$ and that for every $|s| < \frac{1}{r_k}$,

$$\big||\psi_k(\tilde{x}+s)| - |\psi_k(\tilde{x})|\big| \leq \ell_k\big(|\sin(2\pi t_k(\tilde{x}+s))| - |\sin(2\pi t_k\tilde{x})|\big) \leq \ell_k\frac{t_k}{r_k}.$$

Now, to deal with the graph of φ_k, we begin by noticing that

$$|\psi_j'| \leq 2\pi(\ell_j t_j + \ell_j r_j) \leq 4\pi\ell_j r_j.$$

Thus, on each interval $[m/t_k, (m+1)/t_k] \subset \mathbb{T}^1$, the oscillation of φ_{k-1} is at most $\frac{4\pi}{t_k}\sum_{j=1}^{k-1}\ell_j r_j$. Since

$$|\varphi_{k-1}| \leq \sum_{j=1}^{k-1}\ell_j,$$

this proves the lemma. ∎

The next lemma deals with the density of a certain F_k-orbit along the graph of φ_k.

Lemma 16 *The set* $\{F_k^n(0,0)\colon 1 \le n \le q_k\}$ *is* $(\frac{1}{t_k}, \frac{\ell_k t_k}{r_k} + \frac{4\pi}{t_k}\sum_{j=1}^{k-1}\ell_j r_j + \frac{16\pi}{q_k^{2/3}}\sum_{j=1}^{k}\ell_j r_j)$*-dense in the cylinder* $\mathbb{T}^1 \times \mathrm{Ball}\big(0, \ell_k - \sum_{j=1}^{k-1}\ell_j\big)$.

Proof. Since φ_k is an invariant section for F_k, the set of points we are dealing with coincides with

$$\left\{\left(n\frac{s_k p_k}{q_k}, \varphi_k\Big(n\frac{s_k p_k}{q_k}\Big)\right) : 1 \le n \le q_k\right\}. \tag{11}$$

Since p_k and q_k are coprime and $s_k \le 2t_k \le 2q_k^{1/3}$, the projection on the first coordinate of this set consists of at least $q_k^{2/3}/2$ points uniformly distributed on $\mathbb{T}^1$. Therefore, the distance between two consecutive points of the set (11) is less than or equal to

$$\int_{\frac{2j}{q_k^{2/3}}}^{\frac{2(j+1)}{q_k^{2/3}}} \sqrt{1+\varphi_k'(x)^2}\ dx \ \le\ \frac{4\max|\varphi_k'|}{q_k^{2/3}} \ \le\ \frac{16\pi\sum_{j=1}^{k}\ell_j r_j}{q_k^{2/3}},$$

and the claim of this lemma follows from that of the preceding one. ∎

To close the construction, notice that for each $1 \le n \le q_k$, the properties of the inductive construction together with the estimates (10) and $s_{k+1} \le 2q_{k+1}^{1/3} < q_{k+1}^{1/2}$ yield

$$\big|(\rho_{k+1})_n - (\rho_k)_n\big| \le \frac{q_{k+1}^{1/12} q_k q_{k+1}^{1/2}}{q_{k+1}} + C_k q_k|\alpha_{k+1}-\alpha_k| + D_k q_k|\beta_{k+1}-\beta_k| \le \frac{q_k}{q_{k+1}^{5/12}} + \frac{1}{2^{k+1}} \le \frac{1}{2^k}. \tag{12}$$

Letting $n := 1$, this shows that (ρ_k) is a Cauchy sequence, hence it converges to a continuous function $\rho\colon \mathbb{T}^1 \to \mathbb{C}$. Moreover, from Property 3. it follows immediately that (α_k) converges to some angle $\alpha \in [0,1]$. Similarly, by Property 4, (β_k) converges to a certain angle $\beta \in [0,1]$.

Checking that the limit map $F\colon (x,z) \mapsto \big(x+\alpha, e^{2\pi i\beta}z + \rho(x)\big)$ is topologically transitive is not very difficult. Indeed, from Property 2. it follows that

$$\frac{1}{t_k}\sum_{j=1}^{k-1}\ell_j r_j \le \frac{2}{q_k^{1/3}}\sum_{j=1}^{k-1}\ell_j q_j^{1/2} \le \frac{1}{k}.$$

Moreover,

$$\frac{1}{q_k^{2/3}}\sum_{j=1}^{k}\ell_j r_j \le \frac{1}{t_k}\sum_{j=1}^{k-1}\ell_j r_j + \frac{\ell_k r_k}{q_k^{2/3}} \le \frac{1}{k} + \frac{\ell_k}{q_k^{1/6}} = \frac{1}{k} + \frac{1}{q_k^{1/12}}.$$

The last two inequalities combined with Lemma 16 imply that $\{F_k^n(0,0)\colon 1 \le n \le q_k\}$ is $(\varepsilon_k, \delta_k)$-dense in the cylinder $\mathbb{T}^1 \times \mathrm{Ball}\big(0, \ell_k - \sum_{j=1}^{k-1}\ell_j\big)$ for certain sequences (ε_k), (δ_k) converging to zero as k goes to infinite. Since $F_k^n(0,0) = \big(n\alpha_k, (\rho_k)_n(0)\big)$, using (12) and the estimate (valid for $1 \leq n \leq q_k$)

$$\big|n\alpha_k - n\alpha_{k+1}\big| \leq \frac{q_k}{2^{k+2} q_k C_k} \leq \frac{1}{2^{k+1}},$$

we conclude that the orbit of $(0,0)$ under the limit map F is dense in $\mathbb{T}^1 \times \mathrm{Ball}\big(0, \ell_k - \sum_{j=1}^{k-1} \ell_j\big)$ for every $k \in \mathbb{N}$. Finally, since Property 2. yields

$$\sum_{j=1}^{k-1} \ell_j \leq (k-1)\ell_{k-1} = (k-1) q_{k-1}^{1/12} \leq \frac{q_k^{1/12}}{2} = \frac{\ell_k}{2},$$

we have that $\ell_k - \sum_{j=1}^{k-1} \ell_j \geq \frac{\ell_k}{2}$ goes to infinite together with k, thus showing that the F-orbit of $(0,0)$ is dense in the whole space $\mathbb{T}^1 \times \mathbb{C}$.

It remains to show that α and β are rationally independent. Actually, we do not know whether this is always true, but we can ensure it provided that the sequence (q_k) satisfies a supplementary condition.

Lemma 17 *If the sequence (q_k) satisfies*

5. $\sum_{j=k}^{\infty} |\alpha_{j+1} - \alpha_j| + \sum_{j=k}^{\infty} |\beta_{j+1} - \beta_j| < \frac{1}{k q_k}$,

then α and β are rationally independent.

Proof. Since $p_k \leq q_k$, if we consider the representatives of α_k and β_k in $[0,1]$, then we have

$$\alpha_k = \frac{p_k}{q_k}, \quad \beta_k = r_k \alpha_k = \frac{r_k p_k}{q_k} - n_k, \quad \text{where } n_k \in \mathbb{Z}.$$

Assume that $(p,q,r) \neq (0,0,0)$ is 3-tuple of integers such that $p\alpha + q\beta + r = 0$. On the one hand,

$$\begin{aligned} |p\alpha_k + q\beta_k + r| &= |p(\alpha_k - \alpha) + q(\beta_k - \beta) + p\alpha + q\beta + r| \\ &= |p(\alpha_k - \alpha) + q(\beta_k - \beta)| \\ &\leq |p| \sum_{j \geq k} |\alpha_{j+1} - \alpha_j| + |q| \sum_{j \geq k} |\beta_{j+1} - \beta_j|. \end{aligned}$$

On the other hand, if

$$p\alpha_k + q\beta_k + r = p\,\frac{p_k}{q_k} + q\Big(\frac{r_k p_k}{q_k} - n_k\Big) + r$$

equals zero, then

$$p_k(p + q r_k) = q_k(q n_k - r).$$

Since p_k and q_k are coprime, this implies that q_k must divide $p + q r_k$. Nevertheless, since $r_k \leq \sqrt{q_k}$ and $(p,q) \neq (0,0)$, this is impossible for a large-enough k. Therefore, for a large-enough $k \in \mathbb{N}$, the value of $p\alpha_k + q\beta_k + r$ is nonzero, and hence

$$|p\alpha_k + q\beta_k + r| = \left| \frac{p p_k + q(r_k p_k - n_k q_k) + r q_k}{q_k} \right| \geq \frac{1}{q_k}.$$

Therefore,

$$\frac{1}{q_k} \leq |p| \sum_{j \geq k} |\alpha_{j+1} - \alpha_j| + |q| \sum_{j \geq k} |\beta_{j+1} - \beta_j|,$$

which contradicts Property 5. for k larger than $|p|$ and $|q|$. □

3 Reducibility v/s arithmetic properties of the rotation angles

Although the preceding construction provides us with a pair of angles (α, β) that are rationally independent, it also suggests that β is very fast approximated by multiples of α. As we next show, this is the case of every non-reducible smooth cylindrical vortex. What follows is inspired (and may be deduced almost directly) from [13].

We say that a pair $(\alpha, \beta) \in \mathbb{T}^1 \times \mathbb{T}^1$ satisfies a type-1 Diophantine condition, and write $(\alpha, \beta) \in \mathcal{CD}_1$, if there exist $C > 0$ and $\tau \geq 0$ such that for every $n \in \mathbb{Z}$,

$$\left| e^{2\pi i (n\alpha - \beta)} - 1 \right| \geq \frac{C}{n^{1+\tau}}.$$

For a fixed irrational $\alpha \in \mathbb{T}^1$, denote by $\mathcal{CD}_1^\alpha$ the set of $\beta \in \mathbb{T}^1$ such that (α, β) belongs to $\mathcal{CD}_1$. Standard arguments show the next

Lemma 18 *The set $\mathcal{CD}_1^\alpha$ is a countable union of closed sets with empty interior and has full Lebesgue measure.*

The next proposition is nothing but a straightforward application of the classical baby-KAM theorem.

Proposition 19 *If $\rho : \mathbb{T} \to \mathbb{C}$ is a C^∞-function, then for every $(\alpha, \beta) \in \mathcal{CD}_1$ the cylindrical vortex*

$$F : (x, z) \longmapsto \big(x + \alpha, e^{2\pi i \beta} z + \rho(x) \big)$$

is reducible.

Proof. Recall that reducibility is equivalent to the existence of a continuous solution to the cohomological equation

$$\varphi(x + \alpha) - e^{2\pi i \beta} \varphi(x) = \rho(\theta). \tag{13}$$

At the level of Fourier series expansion, this is equivalent to that, for all $n \in \mathbb{Z}$,

$$\hat{\varphi}_n = \frac{\hat{\rho}_n}{e^{2\pi i n \alpha} - e^{2\pi i \beta}},$$

where $\hat{\varphi}_n$ and $\hat{\rho}_n$ stand for the Fourier coefficients of ϕ and ρ, respectively. Since ρ is a C^∞-function, $\hat{\rho}_n$ decreases faster than $1/n^k$ for any $k \geq 1$. The type-1 Diophantine condition on (α, β) allows us to conclude that the coefficients $\hat{\varphi}_n$ defined by the previous equality also decrease faster than $1/n^k$ for any $k \geq 1$. Therefore, they correspond to the coefficients of a C^∞-function which solves our cohomological equation. $\square$

A better result relating the differentiability classes of ρ and the solution φ can be stated as follows:

Proposition 20 [Herman] *Let ρ be a C^r-function. If $(\alpha, \beta) \in \mathcal{CD}_1$ is such that the associated $\tau \geq 0$ satisfies $r > \tau + 1$ and $s := r - 1 - \tau$ is not an integer, then the solution φ to the equation (13) is a C^s-function.*

Finally, concerning the case of "Liouville pairs", we have the following

Proposition 21 *If (α, β) satisfies no type-1 Diophantine condition, then there exists a C^∞-function $\rho : \mathbb{T}^1 \to \mathbb{C}$ such that the equation (13) has no measurable solution. Moreover, ρ can be chosen so that the coefficients $\frac{\hat{\rho}_n}{(e^{2\pi i n\alpha} - e^{2\pi i\beta})}$ do not correspond to those of any distribution.*

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